Version AA/5

EGT3
ENGINEERING TRIPOS PART IIB

Monday 6 May 2024 2 to 3.40

Module 4A15

ACOUSTICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A15 Aeroacoustics data sheet (5 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) The surface $x = 0$ in Fig. 1(a) oscillates with a velocity in the x direction equal to the real part of $U \exp\left[\mathrm{i}(\omega t - k_y y)\right]$. Here U is a complex amplitude, y is the vertical coordinate, k_y is a positive constant, and the other symbols have their usual meanings. The medium in $x > 0$ has density ρ_1 and sound speed c_1, with $\omega/c_1 < k_y$. The pressure field therein is given by

$$p_1' = A(x)\mathrm{e}^{\mathrm{i}(\omega t - k_y y)}.$$

Find $A(x)$. (Hint: p_1' must obey the homogeneous wave equation in $x > 0$.) [40%]

(b) The plane at $x = 0$ in Fig. 1(b) forms a boundary between two media. The densities and sound speeds are (ρ_0, c_0), in $x < 0$, and (ρ_1, c_1), in $x > 0$. In the left-hand medium, a plane wave with complex amplitude I approaches the boundary at an angle θ to the x axis. At the boundary it is reflected with amplitude R, so that the overall pressure is given by

$$p_0' = I\mathrm{e}^{\mathrm{i}\omega(t - x\cos\theta/c_0 - y\sin\theta/c_0)} + R\mathrm{e}^{\mathrm{i}\omega(t + x\cos\theta/c_0 - y\sin\theta/c_0)}.$$

(i) The sound speeds and the angle θ are such that $c_1 \sin\theta > c_0$. Find R/I. [40%]

(ii) What is the mean energy flux into the right-hand medium? In the light of your answer, comment on the ratio R/I. [20%]

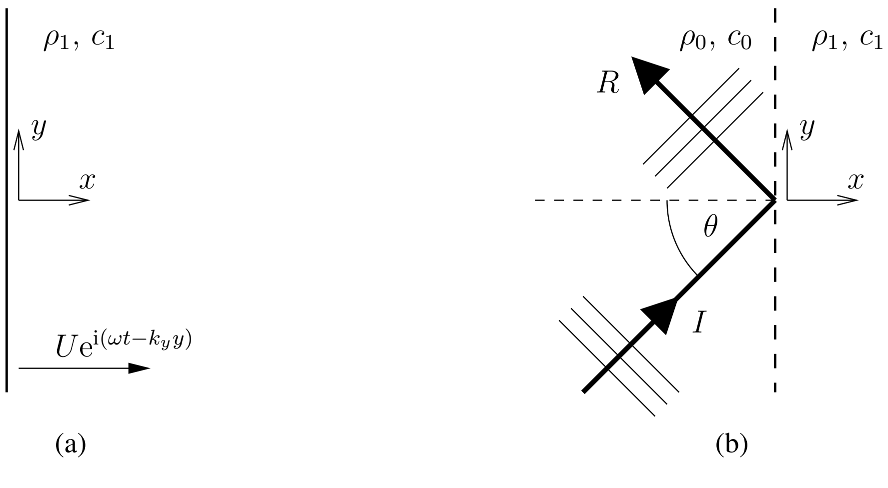

Fig. 1

2 An unsteady force distribution (F_1, F_2, F_3) has length-scale l, time-scale T, and characteristic density ρ_0. The components F_2 and F_3 are both zero, while F_1 is expressed in terms of a function S as

$$F_1(\mathbf{x},t) = \frac{\rho_0 l}{T^2} S\left(\frac{|\mathbf{x}|}{l}, \frac{t}{T}\right).$$

The acoustic density fluctuation $\rho'(\mathbf{x},t)$ due to this source is governed by the equation

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = -\frac{\partial F_1}{\partial x_1},$$

where c_0 is the speed of sound. Symbols not explicitly defined have their usual meanings.

(a) Show that the density fluctuation in the far field can be written

$$\rho'(\mathbf{x},t) = -\frac{1}{4\pi c_0^2 |\mathbf{x}|} \int \frac{\partial}{\partial y_1} F_1\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) \mathrm{d}\mathbf{y}.$$

[20%]

(b) The source is such that $l/c_0 \ll T$. Give a physical interpretation of this relation, and state its implication for further approximation of the integral in part (a). [20%]

(c) On the basis of the integral in part (a) and the assumption $l/c_0 \ll T$, obtain an order-of-magnitude estimate for the far-field density fluctuation. [20%]

(d) An alternative formulation for the acoustic far-field is

$$\rho'(\mathbf{x},t) = -\frac{1}{4\pi c_0^2 |\mathbf{x}|} \frac{\partial}{\partial x_1} \int F_1\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right) \mathrm{d}\mathbf{y}.$$

Given that $l/c_0 \ll T$ still, what order-of-magnitude estimate does this yield? Note that you will need to bring the x_1 derivative inside the integral. [20%]

(e) Identify which of your answers to parts (c) and (d) is incorrect, and explain why. [20%]

3 (a) Explain why a rigid-walled annular duct of inner radius a and outer radius b can be approximated as a rectangular duct when $(b-a)/a \ll 1$. [10%]

(b) What are the appropriate boundary conditions for acoustic waves in the rectangular duct? [10%]

(c) Show that for acoustic disturbances of angular frequency ω these boundary conditions lead to axial wavenumbers

$$k_{mn} = \sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{m\pi}{b-a}\right)^2 - \left(\frac{n}{R}\right)^2}$$

for integers m and n. Here c_0 is the speed of sound and $R = (a+b)/2$. [60%]

(d) Hence show that in the annular duct all modes of circumferential mode number n are "cut-off" at radian frequencies ω less than nc_0/R. [10%]

(e) What condition on fan speed is required if the fundamental rotor-alone tone is to be cut off? [10%]

4 An acoustic liner for low frequency sound absorption consists of a perforated surface. Each hole opens onto a cavity of volume 0.0004 m^3 and together the hole and cavity behave like a Helmholtz resonator. The difference between the pressure perturbation just outside a hole and within its cavity is equal to

$$\rho_0 l \frac{\mathrm{d}u'}{\mathrm{d}t} + \alpha u'$$

where ρ_0 is the mean density, l is 0.6 times the hole diameter, u' is the velocity of the air flowing through the hole, $\alpha = \rho_0 c_0 k$, with $k = 0.1$, and c_0 is the speed of sound. The ambient temperature is 600 K and pressure 1 bar.

(a) What hole size would you choose to absorb sound of frequency 250 Hz? [60%]

(b) With this choice of hole size, determine the rate of absorption of sound energy at 250 Hz by a single hole in terms of the root-mean-square pressure perturbation at the hole. [40%]

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Module 4A15 Aeroacoustics Data Sheet

USEFUL DATA AND DEFINITIONS

Physical Properties

Speed of sound in an ideal gas $\sqrt{\gamma RT}$, where T is temperature in Kelvins

Units of sound measurement

$$\begin{aligned}
\text{SPL (sound pressure level)} &= 20\log_{10}\left(\frac{p'_{rms}}{2\times 10^{-5}\text{Nm}^{-2}}\right)\text{dB} \\
\text{IL (intensity level)} &= 10\log_{10}\left(\frac{\text{intensity}}{10^{-12}\text{watts m}^{-2}}\right)\text{dB} \\
\text{PWL (power level)} &= 10\log_{10}\left(\frac{\text{sound power output}}{10^{-12}\text{watts}}\right)\text{dB}
\end{aligned}$$

Definitions

Surface impedance Z_s, relates the pressure perturbation applied to a surface, p', to its normal velocity v'; $p' = Z_s v'$

Characteristic impedance of a fluid $\rho_0 c_0$

Specific impedance of a surface $Z_s/(\rho_0 c_0)$

Wavenumber $k = \omega/c_0 = 2\pi/\lambda$, where λ is the wavelength

Helmholtz number (or compactness ratio) = kD, where D is a typical dimension of the source.

Strouhal number $= \omega D/(2\pi U)$ for sound of frequency ω (in rad/s), produced in a flow with speed U, length scale D.

Basic equations for linear acoustics

Conservation of mass

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0$$

Conservation of momentum

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0$$

Isentropic

$$c_0^2 = \left.\frac{dp}{d\rho}\right|_S$$

Wave equation

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

Energy density

$$e = \frac{1}{2}\rho_0 v'^2 + \frac{1}{2\rho_0 c_0^2} p'^2$$

Intensity $\mathbf{I} = p'\mathbf{v}'$

Velocity potential ϕ' satisfies the wave equation and $p' = -\rho_0 \frac{\partial \phi'}{\partial t}$, $\mathbf{v}' = \nabla \phi'$.

Autocorrelation $F(\xi)$, the autocorrelation of $f(y)$ is given by

$$F(\xi) = \overline{f(y)f(y+\xi)}$$

$$F(0) = \overline{f^2}$$

Integral length scale, l

$$l\overline{f^2} = \int_{-\infty}^{\infty} F(\xi)d\xi$$

Sound power

Sound power from a source is defined as

$$P = \int_S \bar{\mathbf{I}} \cdot \mathbf{dS} = \int_{S_\infty} \frac{\overline{p'^2}}{\rho_0 c_0}\mathbf{dS}$$

for a statistically stationary source. For an outward propagating spherically symmetrical sound field $P = \frac{\overline{p'^2}}{\rho_0 c_0} 4\pi r^2$, where p' is the acosutic pressure at radius r.

For a sound field, which is a function of spherical polar coordinates r, θ only, and is independent of the azimuthal angle,

$$P = 2\pi r^2 \int_0^{\pi} \frac{\overline{p'^2}}{\rho_0 c_0} \sin\theta d\theta$$

Simple wave fields

1D or plane wave

The general solution of the 1D wave equation is $p'(x,t) = f(t - x/c_0) + g(t + x/c_0)$, where f and g are arbitrary functions. In a plane wave propagating to the right $p' = \rho_0 c_0 u'$; in a plane wave propagating to the left $p' = -\rho_0 c_0 u'$, u' being the particle velocity.

Spherically symmetric sound fields

The general spherically symmetric solution of the 3D wave equation is

$$\phi'(r,t) = \frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r},$$

where r is the distance from the source; f and g are arbitrary functions.

$\cos\theta$ dependence

The general solution of the 3D wave equation with $\cos\theta$ dependence is

$$p'(\mathbf{x},t) = \frac{\partial}{\partial x}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right] = \cos\theta\,\frac{\partial}{\partial r}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right]$$

Useful mathematical formulae

Spherical polar coordinates (r, θ, ψ)

Gradient

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p'}{\partial \psi}\right)$$

Divergence

$$\nabla \cdot \mathbf{v}' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v'_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v'_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v'_\phi}{\partial \psi}$$

Laplacian

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial p'}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial \psi^2}$$

Delta functions

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1D δ-function $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(ax-b) f(x) dx = f(b/a)/|a|$

3D δ-function $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$

Convolution algebra

Convolution of $f(\mathbf{x})$ and $g(\mathbf{x})$

$$(f \star g)(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

Commutative properties

$$f \star g = g \star f$$

$$\frac{\partial}{\partial x_i}(f \star g)(\mathbf{x}) = f \star \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \star g$$

Green's function

3D Green's function for wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right) g(\mathbf{x},t|\mathbf{y},\tau) = \delta(t-\tau)\delta(\mathbf{x}-\mathbf{y})$$

$$g(\mathbf{x},t|\mathbf{y},\tau) = \frac{\delta\left\{|\mathbf{x}-\mathbf{y}| - c_0(t-\tau)\right\}}{4\pi c_0 |\mathbf{x}-\mathbf{y}|}$$

Lighthill's Acoustic Analogy

Lighthill's equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right) \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

For cold, isentropic, low Mach-number jets, T_{ij} can be approximated as:

$$T_{ij} = \rho_0 u_i u_j$$

Lighthill eight power law Acoustic power,

$$P_a \sim \frac{\rho_o d_j^2}{c_0^5} u_j^8,$$

where d_j and u_j are the jet exit diameter and velocity, respectively.

In a cylindrical duct of radius a

The pressure field is given by

$$p'(\mathbf{x},t) = e^{i(\omega t + n\theta)} J_n(z_{mn} r/a)(Ae^{-ikx_3} + Be^{ikx_3}),$$

where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_0^2 - z_{mn}^2/a^2)^{1/2}$.

For large azimuthal wavenumber, n

$$z_{1n} \approx n + 1.85 n^{1/3}$$

In a duct of varying area $A(x)$

Webster horn equation

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{A}\frac{\partial}{\partial x}\left(A \frac{\partial p'}{\partial x}\right) = 0$$

Modified Webster horn equation $\psi(x) = \hat{p}(x) A^{1/2}$, $A = \pi a^2$

$$\frac{d^2\psi}{dx^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{dx^2}\right)\psi = 0$$