a) Wave eqn: $\frac{1}{c_1^2}\frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = 0$

Substitute for p_1: $\left(k_y^2 - \frac{\omega^2}{c_1^2}\right) A - \frac{d^2 A}{dx^2} = 0$

Hence $A = a_1 e^{\gamma x} + a_2 e^{-\gamma x}$; $\gamma^2 = k_y^2 - \frac{\omega^2}{c_1^2}$

A remains finite as $x \to \infty \Rightarrow A = a_2 e^{-\gamma x}$

At $x = 0$ $\rho_1 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$

$$i\omega\rho_1 U = \gamma a_2$$

$$\Rightarrow A = \frac{i\omega\rho_1 U}{\gamma} e^{-\gamma x}$$

b) (i) Let the horizontal velocity generated at $x=0$ be $U e^{i(\omega t - k_y y)} = U e^{i\omega(t - y\sin\theta/c_0)}$. Then we have

$$p_1 = A(x) e^{i(\omega t - k_y y)}$$ as in part (a), so

$$I + R = \frac{i\omega\rho_1 U}{\gamma}$$ equating pressures

$$(I - R)\frac{\cos\theta}{\rho_0 c_0} = U$$ equating horizontal velocities

Eliminate U: $I + R = \frac{i\omega\rho_1\cos\theta}{\rho_0 c_0 \gamma}(I - R)$

Define $\chi = \dfrac{\omega \rho_1 \cos\theta}{\rho_0 c_0 \gamma}$, then

$$I(1 - i\chi) = -R(1 + i\chi)$$

$$\frac{R}{I} = -\frac{1 - i\chi}{1 + i\chi}$$

(ii) Energy flux (mean) $= \frac{1}{2} \mathrm{Re}(pu^*)$

Easiest @ $x = 0$: $p = \dfrac{i\omega\rho_1 U}{\gamma} e^{-i\omega y \sin\theta / c_0}$

$$u = U e^{-i\omega y \sin\theta / c_0}$$

$\Rightarrow$ pu^* is imaginary, and flux $= 0$

Given this answer, we expect all incoming energy in $x < 0$ to be reflected, i.e. $|R|^2 = |I|^2$. Calculating $|R/I|$ gives unity, confirming $|R|^2 = |I|^2$.

a) Solution for ρ' is $g * \left(-\frac{\partial F_i}{\partial x_i}\right)$, where g is the Green's function;

$$g(\underline{x}, t; \underline{y}, \tau) = \frac{\delta(|\underline{x}-\underline{y}| - c_0(t-\tau))}{4\pi c_0 |\underline{x}-\underline{y}|}$$

So

$$\rho'(\underline{x}, t) = -\frac{1}{4\pi c_0} \int \frac{\partial F}{\partial y_i} \frac{\delta(|\underline{x}-\underline{y}| - c_0(t-\tau))}{|\underline{x}-\underline{y}|} \, d\underline{y} \, d\tau$$

$$= -\frac{1}{4\pi c_0^2} \int \frac{\partial F_i}{\partial y_i}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) \frac{d\underline{y}}{|\underline{x}-\underline{y}|}$$

Now $|\underline{x}-\underline{y}| \simeq |\underline{x}|$ in far field. Hence, noting $F_2 = F_3 = 0$,

$$\rho'(\underline{x}, t) \simeq -\frac{1}{4\pi c_0^2 |\underline{x}|} \int \frac{\partial F_1}{\partial y_1}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) d\underline{y}$$

b) The time interval L/c_0 is the typical difference in emission times from different parts of the source. The relation states that this interval is much less than the time scale over which the source changes, i.e. that all the emission times contributing to $\rho'(\underline{x}, t)$ can be taken equal to a good approximation. This means we can simplify the retarded time:

$$t - \frac{|\underline{x}-\underline{y}|}{c_0} \simeq t - \frac{|\underline{x}|}{c_0}.$$

c) Define $S_x(\tilde{\underline{x}}, \tilde{t}) = \frac{\partial}{\partial \tilde{x}} S(\tilde{\underline{x}}, \tilde{t})$ (dimensionless, like S)

Thus $\frac{\partial}{\partial y_1} F_1\left(\underline{y}, t - \frac{|\underline{x}|}{c_0}\right) = \frac{\rho_0 l}{T^2} \cdot \frac{1}{l} S_x\left(\frac{\underline{y}}{l}, \frac{t - |\underline{x}|/c_0}{T}\right)$

$\uparrow$ $\frac{\partial (y_1/l)}{\partial y_1}$

$\sim \frac{\rho_0}{T^2}$

Hence $\rho' \sim \frac{1}{c_0^2 |\underline{x}|} \frac{\rho_0}{T^2} l^3 = \rho_0 \left(\frac{l}{c_0 T}\right)^2 \frac{l}{|\underline{x}|}$

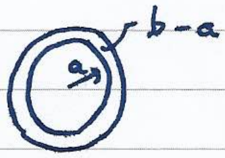
d) Define $S_t = \frac{\partial}{\partial \tilde{t}} S(\tilde{\underline{x}}, \tilde{t})$

then $\frac{\partial}{\partial x_1} F_1\left(\underline{y}, t - \frac{|\underline{x}|}{c_0}\right) = \frac{\rho_0 l}{T^2} S_t\left(\frac{\underline{y}}{l}, \frac{t - |\underline{x}|/c_0}{T}\right) \frac{\partial}{\partial x_1}\left(\frac{t - |\underline{x}|/c_0}{T}\right)$

$\sim \frac{\rho_0 l}{T^2} \frac{1}{c_0 T}$

Thus $\rho' \sim \frac{1}{c_0^2 |\underline{x}|} \cdot \frac{\rho_0 l}{c_0 T^3} \cdot l^3 = \rho_0 \left(\frac{l}{c_0 T}\right)^3 \frac{l}{|\underline{x}|}$

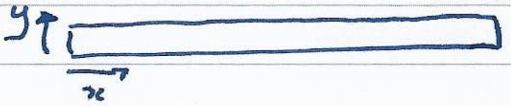
e) Part (c) answer is incorrect, because integration over y_1 gives $\rho'(\underline{x},t) = 0$ (since F_1 is zero outside a restricted region in space). This is consistent with part (d) estimate being smaller; we have no reason to doubt the part (c) expression for $\rho'(\underline{x},t)$, nor the order-of-magnitude estimation process, so the only way we can avoid a contradiction is if the constant of proportionality is zero. Equivalently, to avoid this perfect cancellation it is necessary to recognize the differences in emission times, small though they are. Hence the formulation is unsuitable for the constant-retarded-time assumption.

a) 

Consider slice $\delta\theta$: 

Small quadrilateral element is very close to rectangular when $\frac{b-a}{a} \ll 1$.

Hence dynamics etc close to those for rectangular duct; i.e. curvature effect negligible.

b) 

$$\frac{\partial\phi}{\partial y} = 0 \quad @ \; y = 0,\; b-a$$

$$\phi(x = 2\pi a) = \phi(x = 0)$$

c) Disturbances of frequency ω governed by:

$$(\nabla^2 + k^2)\phi = 0 \qquad k = \omega / c_0$$

Note you can obtain the solution via separation of variables and then apply the BCs.

2!

To satisfy B.C.s, $\phi = \sum_{m,n} F(z) \cos\frac{m\pi y}{b-a} \cos\left[\frac{n\pi x}{2\pi a} + \theta\right]$

$$\text{so} \quad \sum_{m,n}\left[\frac{F''(z)}{F(z)} - \left(\frac{m\pi}{b-a}\right)^2 - \left(\frac{n}{a}\right)^2 + k^2\right] F(z)\cos\frac{m\pi y}{b-a}\cos[\;\;] = 0$$

Extract individual (m,n) modes by multiply / integrate to yield

$$F''(z) + \left[k^2 - \left(\frac{m\pi}{b-a}\right)^2 - \left(\frac{n}{a}\right)^2\right] F(z) = 0$$

with sol'ns $F(z) \sim e^{\pm i k_{ax} z}$,

$$k_{ax}^2 = k^2 - \left(\frac{m\pi}{b-a}\right)^2 - \left(\frac{n}{a}\right)^2 \qquad \text{QED ish}$$

d) If $\omega < nc_0/R$,

$$k_{ax}^2 < \frac{n^2}{R^2} - \left(\frac{m\pi}{b-a}\right)^2 - \frac{n^2}{R^2} = -\left(\frac{m\pi}{b-a}\right)^2$$

i.e. $k_{ax}^2 < 0$; mode non-propagating.

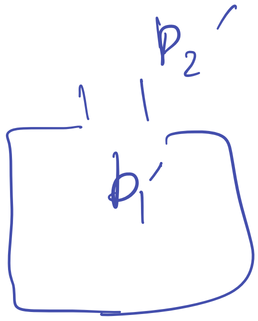
e) Rotor-alone noise has fundamental freq ΩN where N is no. of blades.

and circumferential order $n = N$

Using condition in (d), $\Omega N < \frac{N c_0}{R}$

$$\Omega < c_0/R$$

4.



$$p_2' - p_1' = \rho_0 l \frac{du'}{dt} + \alpha u'$$

$$p_1' = c_0^2 \rho_1'$$

$$\dot{m} = \rho_0 A u' = V \frac{\partial \rho_1'}{\partial t}$$

$$\frac{\partial}{\partial t} \leftrightarrow i\omega$$

$$\frac{\rho_0 A u'}{i\omega V} = \rho_1'$$

$$\Rightarrow \quad p_2' = (\rho_0 l i\omega + \alpha) u' + c_0^2 \frac{\rho_0 A u'}{i\omega V}$$

$$= u' \left\{ \rho_0 l i\omega + \frac{c_0^2 \rho_0 A}{i\omega V} + \alpha \right\}$$

(a) Sound absorption $\propto \overline{p_2' u'}$

$\Rightarrow$ we want large u'

Choose A, so that the Helmholtz resonance freq. is 250 Hz.

$$\rho_0 l i\omega + \frac{c_0^2 A \rho_0}{V i\omega} = 0 \quad \text{when} \quad \omega_0 = 2\pi \times 250$$

$$l = 0.6d, \quad A = \frac{\pi d^2}{4}$$

$p_0 = 1$ bar, $T = 600$ K $\Rightarrow$ $\rho_0 = 0.581$ kg m^{-3}

$c_0 = 491$ m/s

$\Rightarrow d = 3.13$ mm.

(b) $\mathbb{P}$ Sound power absorption $= \overline{p_2' u' A}$

At 250 Hz,

$$p_2' = u' \alpha$$

$$\therefore \mathbb{P} = \frac{\overline{p_2'^2} A}{\alpha}$$

$$= \frac{p_{rms}^2 \pi d^2}{0.4 \rho_0 c_0} = 2.7 \times 10^{-7} p_{rms}^2$$

(in Watts)