Qu.1 Write $p'(r,t) = \frac{A}{r} e^{i\omega(t-r/c_0)}$

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r} = +A\left(\frac{1}{r^2} + \frac{i\omega}{rc_0}\right) e^{i\omega(t-r/c_0)}$$

On $r = a$, $\rho_0 i\omega u_a e^{i\omega t} = A\left(\frac{1}{a^2} + \frac{ik}{a}\right) e^{i\omega(t-a/c_0)} = \frac{A}{a^2}(1+ika) e^{i\omega(t-a/c_0)}$

Hence $A = \frac{\rho_0 i\omega u_a a^2 e^{i\omega a/c_0}}{1+ika} = \rho_0 c_0 a \frac{ika}{1+ika} e^{ika}$

giving $p'(r,t) = \rho_0 c_0 u_a \frac{a}{r} \frac{ika}{1+ika} e^{i\omega(t-(r-a)/c_0)}$

b) When $ka \gg 1$ $p'(r,t) = \rho_0 c_0 u_a \frac{a}{r} e^{i\omega t - ik(r-a)}$

When $ka \ll 1$ $p'(r,t) = \rho_0 c_0 u_a \frac{a}{r} ika\, e^{i\omega t - ik(r-a)}$

a small factor ka smaller than the high frequency form.

c) With the headphones in our ears we hear the near field. The pressure is then related to u_a in a way that is not spherically symmetric and it related to the ~~Incompressible~~ 1D ~~How of~~ How pressure being caused by vibration of a piston in which $p' \sim \rho_0 c_0 u_a'$

Qu 2 $\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)p' = A e^{-\sigma x_1^2}\delta(x_2)\,\delta(x_3)\,e^{i(\omega t - \alpha x)}$

Using the Green function on the data card

$$p'(\underset{\sim}{x},t) = \int d^3y \frac{\delta(|\underset{\sim}{x}-\underset{\sim}{y}| - c_0(t-\tau))}{4\pi c_0|\underset{\sim}{x}-\underset{\sim}{y}|} A e^{-\sigma y_1^2}\delta(y_2)\,\delta(y_3)\,e^{i(\omega(t-\tau) - \alpha y_1)} d\tau$$

$$= A\int \frac{e^{-\sigma y_1^2} e^{i\omega(t - \frac{|\underset{\sim}{x}-\underset{\sim}{y}|}{c_0} - \alpha y_1)}}{4\pi c_0^2|\underset{\sim}{x}-\underset{\sim}{y}|} dy_1$$

where now $\underset{\sim}{y} = (y_1, 0, 0)$

In far-field $|\underset{\sim}{x}-\underset{\sim}{y}| = |\underset{\sim}{x}| - \frac{\underset{\sim}{x}\cdot\underset{\sim}{y}}{|\underset{\sim}{x}|} + O\left(\frac{y^2}{|\underset{\sim}{x}|^2}\right) = |\underset{\sim}{x}| - \cos\theta\, y_1$

$$\frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|} \approx \frac{1}{|\underset{\sim}{x}|} + O\left(\frac{|y_1|}{|\underset{\sim}{x}|}\right)$$

Hence $p'(\underset{\sim}{x},t) = \frac{A e^{i\omega(t-|\underset{\sim}{x}|/c)}}{4\pi c_0^2|\underset{\sim}{x}|}\int e^{-\sigma y_1^2 + i\omega\cos\theta\, y_1/c - i\alpha y_1} dy_1$

$$= \frac{A e^{i\omega(t-|\underset{\sim}{x}|/c)}}{4\pi c_0^2|\underset{\sim}{x}|}\int_{-\infty}^{\infty} e^{-\sigma y_1^2 + i(k\cos\theta - \alpha)y_1} dy_1$$

$$= \frac{A e^{i\omega(t-|\underset{\sim}{x}|/c)}}{4\pi c_0^2|\underset{\sim}{x}|}\sqrt{\frac{\pi}{\sigma}}\; e^{-\frac{(k\cos\theta-\alpha)^2}{4\sigma}}$$ using hint 2.

b) Phase speed of the wave packet is ~~α~~ ~~α~~ ω/α
write wave packet phase as $e^{i\omega(t - \frac{\alpha}{\omega}x_1)}$

phase speed $= \omega/\alpha$.

c) If $\sigma \ll 1$ $\quad e^{-\frac{(k\cos\theta - \alpha)^2}{4\sigma}}$ is exponentially very small

unless $k\cos\theta \approx \alpha$

$\cos\theta \sim \frac{\alpha}{k} = \frac{\alpha c_0}{\omega}$

~~So~~ This is only true ~~if~~ for some θ if $\frac{\alpha c_0}{\omega} \le 1$

i.e if $c_0 \le \omega/\alpha$

i.e. if the wave packet travel supersonically.
Then it sends out a beam in the direction $\theta_0 = \cos^{-1}(\alpha/k)$.

Qu3

i) $\dot{p}_2' = -\frac{c_0^2}{V} m'(t)$ arises from a mass flow rate out of the bulb causing a rate of change of density in the bulb which for an isentropic fluctuation then gives this relationship for the rate of change of pressure. For it to be true we require:

1) the pressure in the bulb is uniform, i.e. p_2' is a function of t only. This requires that the linear size of the bulb is much smaller than c_0/ω and that the fluid in the bulb has negligible inertia which requires that the cross-sectional area of the bulb is much larger than that of the neck.

2) the perturbations need to satify $p_2' = c_0^2 \rho_2'$. This in term requires that the perturbations are linear and isentropic, i.e. that there is negligible heat transfer and irreversible effects ~~like~~ friction.

3) $m'(t)$ needs to be the mass flow out of the bulb as well as the mass flow out of the neck. This requires negligible mass storage in the neck, i.e. the neck is short compared with c_0/ω.

$p_2' - p_1' = \frac{l}{A}\dot{m}$ arises from momentum balance across the neck. It requires

1) Linear perturbations
2) p_2' and p_1' uniform over the cross-section,
3) No viscous forces
4) the fluid in the neck all moves with the same speed which requires negligible boundary layers and negligible compressible mass storage in the neck, i.e. the neck is short compared with c_0/ω (as in 3) above).

ii) For frequency ω $\quad p_2' = -\frac{c_0^2}{Vi\omega} m'$

$$p_2' - p_1' = \frac{l i\omega}{A} m'$$

Eliminating p_2' between the two equations gives

$$-\frac{c_0^2}{Vi\omega} m' - p_1' = \frac{l i\omega}{A} m'$$

$$p_1' = -\left(\frac{c_0^2}{Vi\omega} + \frac{l i\omega}{A}\right) m' = \frac{l}{Ai\omega}(\omega^2 - \omega_0^2) m'$$

where $\omega_0^2 = \frac{c_0^2 A}{Vl}$.

iii) For frequency ω

$$p'(r,t) = \frac{i\omega}{4\pi r} m'(t - r/c_0)$$

$$= \frac{i\omega}{4\pi r} \frac{Ai\omega}{l(\omega^2 - \omega_0^2)} p_1'(t - r/c_0)$$

Hence $$p'(r,t) = -\frac{A}{4\pi r l}\,\frac{\omega^2}{\omega^2-\omega_0^2}\,p_1'(t-r/c_0)$$

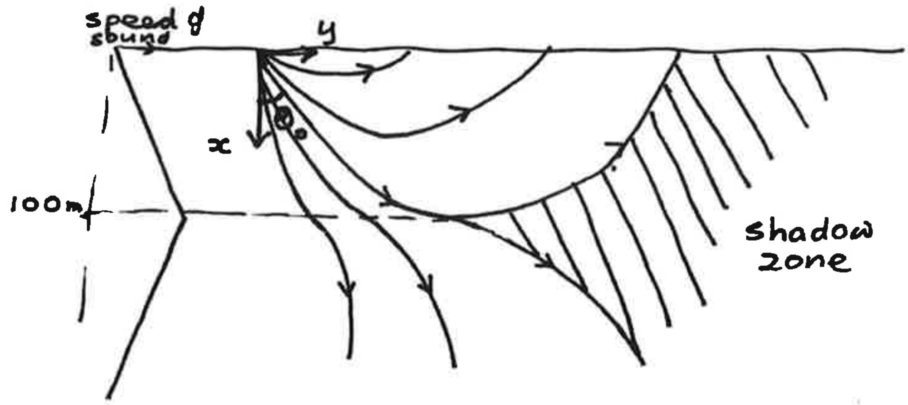
iv) The velocity perturbation in the neck becomes very large at the resonant frequency and this leads to an additional damping term in which $p_2' - p_1' = \frac{l}{A}\dot{m} + \alpha m$. This term is caused by friction, heat transfer, the radiation of acoustic energy to infinity and nonlinear effects, as the kinetic energy in the jet flow in the jet emerging from the neck is dissipated.

Q4 a) Snell's law states that $\dfrac{\sin\theta}{C_0(x)}$ is constant along a ray.

Hence in $0<x<100$m, the angle between a ray and the x-axis increases with depth.

in $x>100$m, the angle between the ray and the x-axis decreases with depth.

The ray pattern we would expect to see is therefore as sketched below and a shadow zone is formed

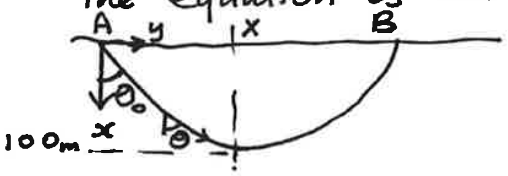


b) (i) The ray that forms the boundary of the shadow zone is horizontal at $x=100$m. Hence is angle θ_0 between the vertical axis and the ray at the surface is given by

$$\frac{\sin\theta_0}{1450} = \frac{\sin 90^\circ}{1500}$$

$$\sin\theta_0 = \frac{145}{150} \Rightarrow \quad \underline{\underline{\theta_0 = 75.16^\circ}}$$

(ii) To find the horizontal distance from the source at which this ray again meets the surface, we need to find the equation of the ray.



From symmetry, $AB = 2\times AX$

Write the speed of sound in $0<x<100$m as $c_0(x) = c_0(0)(1+\alpha$

where $c_0(0) = 1450$, $c_0(0)\alpha = \dfrac{50}{100} \quad \Rightarrow \quad \alpha = \dfrac{0.5}{1450}\ m^{-1}$

Along this ray $\dfrac{\sin\theta}{C_0(x)} = \dfrac{\sin\theta_0}{C_0(0)}$.

Hence $\sin\theta = (1+\alpha x)\sin\theta_0$

$$\frac{dy}{dx} = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(1+\alpha x)\sin\theta_0}{\left(1-\sin^2\theta_0(1+\alpha x)^2\right)^{1/2}}$$

$$AX = \int_0^{100} \frac{(1+\alpha x)\sin\theta_0}{\left(1-\sin^2\theta_0(1+\alpha x)^2\right)^{1/2}}\,dx$$

$$= -\left[\frac{1}{\alpha\sin\theta_0}\left(1-\sin^2\theta_0(1+\alpha x)^2\right)^{1/2}\right]_0^{100}$$

$$= \frac{-1}{\alpha\sin\theta_0}\left[\left(1-\sin^2\theta_0(1+\alpha\, 100)^2\right)^{1/2} - \cos\theta_0\right]$$

$$= \frac{\cos\theta_0}{\alpha\sin\theta_0}$$

since $\sin\theta_0(1+\alpha 100) = \sin\theta_0\left(1+\frac{50}{1450}\right)$

$= \sin\theta_0 \frac{1500}{1450}$

$= 1$

$$= \frac{1450}{0.5}\cot\theta_0$$

$$= 768\,\text{m}$$

The horizontal distance from the source where the ray again meets the surface is $2\times AX = \underline{\underline{1536\,\text{m}}}$