4A15 Examiner's comments:
Q1 Oscillating sphere

The question was answered well by most candidates. It is similar to an example problem. The difference is that the boundary condition is on pressure instead of velocity. This confused some candidates.

Q2 Helmholtz resonator

A popular question. Most candidates did not take the effective length into account for part (b), and got the resonance frequency wrong by an order of magnitude. Almost all students got part (c) wrong. It turns out that the dimensions of the coke can are not really much smaller than the acoustic wavelength (an assumption that most candidates listed as an answer to part (a)), so the Helmholtz formula is not accurate here.

Q3 Multipole expansion

The least popular question, with a wide range of responses. Some were unfamiliar with multipole expansion. Some failed to realise that the strength of a dipole must be a vector. Several failed to use the correct Greens function for the Helmholtz equation, instead trying to use the Greens function for the wave equation. It is possible to derive the Greens function for the Helmholtz equation from that for the wave equation, but only one candidate succeeded in this.

Q4 Webster horn equation

A reasonably popular question. Some failed with bookwork to derive the Webster horn equation. Quite a few failed to spot that 0.5(1+cos(2*a*x)) = cos(a*x)^2, but succeeded otherwise with part (b). This affected their ability to come up with a simple answer for part (c), but the principle was mostly understood. Quite a few failed to notice that if alpha is taken to be imaginary, then the shape of the horn becomes cosh(). (d) was mostly done correctly. Most candidates discussed the tuning of the harmonic series for a bugle in party (e)

Q1. (a) Assuming that the pressure field ~~takes~~ takes the form

$$p'(\underline{x},t) = \frac{\partial}{\partial x}\left(\frac{f(r-c_0 t)}{r}\right)$$

We have assumed this form because $\frac{f(r-c_0t)}{r}$ satisfies an outgoing wave solution to the wave equation. Therefore $\frac{\partial}{\partial x}\left(\frac{f(r-c_0t)}{r}\right)$ must also be a solution to the wave equation.

Let $f(r-c_0t) = \alpha e^{\frac{i\omega}{c_0}(c_0t-r)}$,

$$= e^{i(\omega t - kr)}$$

$$\therefore p'(\underline{x},t) = \frac{\partial}{\partial x}\left\{\alpha\frac{e^{i(\omega t-kr)}}{r}\right\}$$

Here it is assumed that the

physical pressure field will be the real part of this expression.

$(k = \omega / c_0)$

$$p'(r,\theta,t) = \alpha \left\{ -\frac{1}{r^2} - \frac{ik}{r} \right\} e^{i(\omega t - kr)} \times \frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

$$\therefore \quad p'(r,\theta,t) = \alpha \left\{ -\frac{1}{r^2} - \frac{ik}{r} \right\} e^{i(\omega t - kr)} \cos\theta \qquad -(1)$$

The BC at $r = a$ can be written as

$$p'(r=a,\theta,t) = \mathrm{Re}\left\{ A \cos\theta \, e^{i\omega t} \right\} \qquad -(2)$$

Substituting $r = a$ in Eq (1) and equating it to the RHS in (2):

$$\alpha\left\{-\frac{1}{a^2}-\frac{ik}{a}\right\}e^{i(\cancel{\omega t}-ka)}\cancel{\cos\theta}$$

$$= A\,\cancel{\cos\theta}\,\cancel{e^{i\omega t}}$$

$$\Rightarrow \alpha = \frac{-A e^{ika} a^2}{(1+ika)} \qquad -(3)$$

It is easier to calculate the radiated power in the farfield.

In the far fied,

$$\frac{p'}{u'} \sim \rho_0 c_0.$$

$$\therefore u' = \frac{p'}{\rho_0 c_0} \qquad -(4)$$

Also, as $r \to \infty$,

$$\frac{1}{r^2} \ll \frac{k}{r}$$

$\therefore$ Eq (1) simplifies to

$$p'(r,\theta,t) \sim -\frac{ik\alpha}{r} e^{i(\omega t - kr)} \cos\theta \qquad -(5)$$

Intensity $I = p'u'$ $\qquad -(6)$

$$\bar{I} = \frac{1}{2} \mathrm{Re}\left\{ \hat{p}\,\hat{u}^* \right\} \qquad -(7)$$

From (5), (6), (7) & (4)

$$\bar{I} = \frac{1}{2} \mathrm{Re}\left\{ \frac{k^2|\alpha|^2}{r^2 \rho_0 c_0} \cos^2\theta \right\}$$

$$\text{Power} = \int \bar{I}\, ds$$

$$= \int_0^{\pi} \bar{I}\, 2\pi r^2 \sin\theta\, d\theta$$

$$= \frac{k^2|\alpha|^2 \cdot 2\pi \cancel{r^2}}{2\, \cancel{r^2} \rho_0 \omega} \underbrace{\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta}_{\text{Int}}$$

$$\text{Int} = \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \quad -(8)$$

Let $-\cos\theta = s$

$$\therefore \text{Int} = \int_{-1}^{1} s^2\, ds = \left[\frac{s^3}{3}\right]_{-1}^{1}$$

$$= \frac{2}{3}$$

$\therefore$ Eq (8) becomes

$$\text{Power} = \frac{k^2}{3} \frac{|A|^2 \, 2\pi}{\rho_0 c_0}$$

Using Eq (3),

$$\text{Power} = \frac{2\pi}{3} \frac{k^2}{\rho_0 c_0} \frac{A^2 a^4}{(1+k^2a^2)}$$

$$\boxed{\text{Power} = \frac{2\pi}{3} \frac{a^2 A^2}{\rho_0 c_0} \frac{k^2 a^2}{(1+k^2a^2)}} \quad -(9)$$

(b) $\lambda \gg a \Rightarrow ka \ll 1$

(compact source)

& $\lambda \ll a \Rightarrow ka \gg 1$

From (a).

Power $(ka \gg 1)$

$$\sim \frac{2}{3}\pi \frac{a^2A^2}{\rho_0 c_0}$$

& Power $(ka \ll 1)$

$$\sim \frac{2}{3}\pi \frac{a^2A^2}{\rho_0 c_0}(ka)^2.$$

Comparing those two expressions we see that to get the same power A needs to be higher by a factor of $\frac{1}{(ka)}$ ~~when~~

When $ka \ll 1$ to get the same power when $ka \gg 1$.

Eg If $ka = 0.1$, A needs to be 10 times larger to radiate the same power as when $ka = 10$.

Q2

(a) . Narrow opening connected to a large cavity

- Small dimensions of the neck & cavity relative to the acoustic wavelength
- Viscous damping is negligible
- Perturbations are linear and isentropic

(b) $A = 4 \times 10^{-4}\ m^2$

$\pi a^2 = A \Rightarrow a = \sqrt{\frac{A}{\pi}} = 1.1 \times 10^{-2}\ m$

The effective length of the neck

$l' \approx l + 1.2a \sim 1.2a \quad (l << a)$

$V = 0.33\ L = 0.33 \times 10^{-3}\ m^3$

$$\therefore f = \frac{c_0}{2\pi}\sqrt{\frac{A}{l'V}} = \frac{347}{2\pi}\sqrt{\frac{4\times10^{-4}}{1.2\times1.1\times10^{-2}\times0.33\times10^{-3}}}$$

$$f = \underline{529.2\ Hz}$$

(c) For $f = 529.2$ Hz, the wavelength

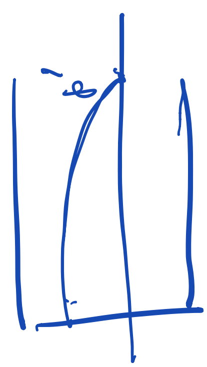
$$\lambda = \frac{347}{529.2} = 0.66 \text{ m}$$

The can has a height of 0.1 m.

$\therefore$ the wavelength is not much larger that the size of the cavity.

For a given ~~a~~ wavelength, a larger cavity has a lower stiffness compared to a smaller cavity. The lower stiffness would result in a lower resonance frequency

(d)



We now have a duct that is closed at ~~a~~ one end & open at the other

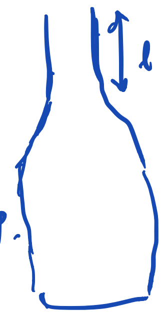
It acts as a quarter-wavelength oscillator. Thus,

$$h = \lambda/4$$

or $\lambda = 4h = 0.4$ m

$$f = \frac{c}{\lambda} = \frac{347}{0.4} = \underline{867.5 \text{ Hz}}$$

(e) The length of the 'neck' of a bottle is much larger than the length of the can. This explains the drop in frequency.



(f) It could be achieved by either:

(i) Increasing the effective length of the neck for example, by inserting a rolled sheet of paper. But note paper is not rigid and will provide some damping.

(ii) Reducing the area of the opening

(iii) Reducing the speed of sound by using a denser gas.

Multipole Questions

$$\hat{p} = \sum_n \hat{S}_n f(x - x_n) \qquad f(x) = \frac{1}{|x|} e^{-ik|x|}$$

a) $$f(x - x_n) = f(x) - x_n \cdot \nabla f(x) + \dots$$

$$\hat{p} = \left(\sum \hat{S}_n\right) f(x) - \left(\sum S_n x_n\right) \cdot \nabla f(x) + \dots$$

b) $\sum \hat{S}_n = 0$ but higher order terms are non-zero means that there will be significant directional dependence in the far field.

c) $\sum \hat{S}_n x_n = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 10 \end{pmatrix}$

the significance of higher order terms in the multipole expansion increases as frequency does, so using just the dipole term will become less accurate as the frequency increases.

d) $$\hat{p} = \int \frac{e^{-ik|x-y|}}{4\pi|x-y|} 4\pi S(y)\, dy$$

$$= \int \frac{e^{-ik|x-y|}}{|x-y|} S(y)\, dy$$

$$= \int f(x-y)\, S(y)\, dy$$

e) $f(x-y) = f(x) - y \cdot \nabla f(x) + \dots$

$$\hat{\rho} = \underbrace{\left[\int S(y)\,dy\right]}_{\text{monopole}} f(x) - \underbrace{\left[\int S(y)\,y\,dy\right]}_{\text{dipole}} \cdot \nabla f(x) + \dots$$

f)

$$\hat{p}(r,\theta) = -\frac{1}{2} i\omega\rho_0 U a^3 \cos\theta \frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right)$$

$$= -\frac{1}{2} i\omega\rho_0 U a^3 \frac{\partial}{\partial x_1}\left(\frac{e^{-ikr}}{r}\right)$$

$$= -\frac{1}{2} i\omega\rho_0 U a^3 \, \underline{e}_1 \cdot \nabla\left(\frac{e^{-ik|\underline{x}|}}{|\underline{x}|}\right)$$

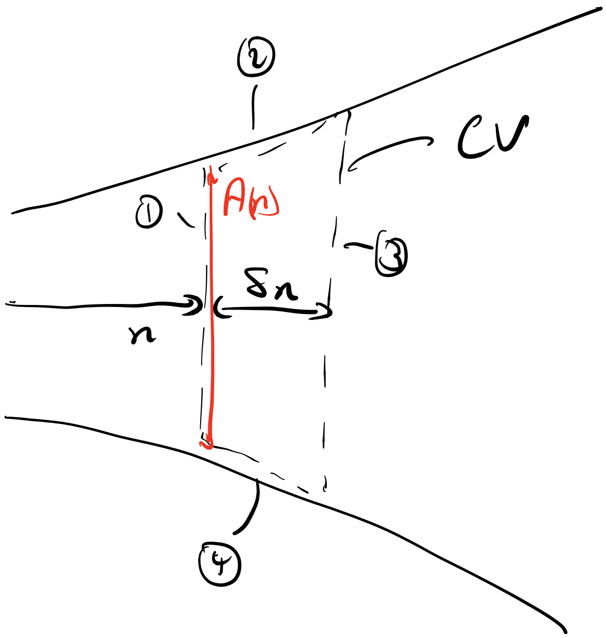
hence monopole strength is zero

dipole strength is

$$\underline{\underline{\frac{1}{2} i\omega\rho_0 U a^3 \, \underline{e}_1}}$$

Webster Horn Question

a)



$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0$$

$$\int_{CV} \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\, dV - \int_{CV} \nabla^2 p \, dV = 0$$

$$\int_{CV} \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\, dV - \int_{CS} n \cdot \nabla p \, dS = 0$$

on 2 & 4 $n \cdot \nabla p = 0$ because of the wall

on 1 $\quad n \cdot \nabla p = -\frac{\partial p}{\partial x}$

on 3 $\quad n \cdot \nabla p = \frac{\partial p}{\partial x}$

assume $p, \frac{\partial p}{\partial x}, \dots$ are constant over cross-section

$$\int_{cv} \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \, dV + \left.\frac{\partial p}{\partial x}\right|_x A(x) - \left.\frac{\partial p}{\partial x}\right|_{x+\delta x} A(x+\delta x) = 0$$

let $\delta x \to 0$

$$\frac{A \delta x}{c^2} \frac{\partial^2 p}{\partial t^2} - \delta x \frac{\partial}{\partial x}\left(A \frac{\partial p}{\partial x}\right) = 0$$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{A}\frac{\partial}{\partial r}\left(A\frac{\partial p}{\partial r}\right) = 0$$

b) $A = \frac{A_0}{2}\left(1 + \cos(2\alpha x)\right)$

$$= A_0 \cos^2(\alpha x) \quad = \pi a^2$$

$$a = \sqrt{\frac{A_0}{\pi}} \cos(\alpha x)$$

$$\frac{da}{dx} = -\sqrt{\frac{A_0}{\pi}}\, \alpha \sin(\alpha x)$$

$$\frac{d^2a}{dx^2} = -\sqrt{\frac{A_0}{\pi}}\, \alpha^2 \cos(\alpha x)$$

$$\frac{1}{a}\frac{d^2a}{dx^2} = -\sqrt{\frac{A_0}{\pi}}\, \alpha^2 \cdot \sqrt{\frac{\pi}{A_0}}$$

$$= -\alpha^2$$

$$\frac{d^2\psi}{dx^2} + \left(k^2 + \alpha^2\right)\psi = 0$$

$$\psi = I e^{-i\sqrt{k^2+\alpha^2}\,x} + R e^{i\sqrt{k^2+\alpha^2}\,x}$$

$$\hat{p} = A^{-1/2}\psi$$

$$= A_0^{-1/2}\cos^{-1}(\alpha x)\ \psi$$

$$= A_0^{-1/2} I \frac{e^{-i\sqrt{k^2+\alpha^2}\,x}}{\cos(\alpha x)} + A_0^{-1/2} R \frac{e^{i\sqrt{k^2+\alpha^2}\,x}}{\cos(\alpha x)}$$

c) for no propagation we need

$$k^2 + \alpha^2 < 0$$

take α to be imaginary, $\alpha = i\beta$

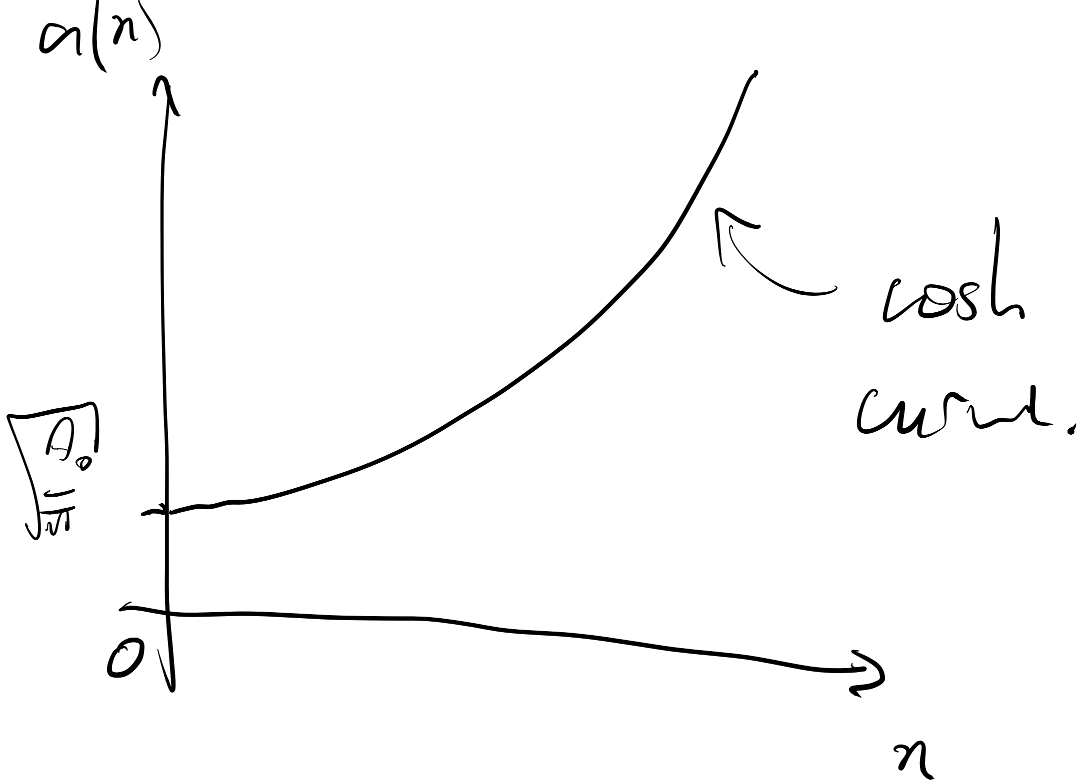
$$k^2 - \beta^2 < 0$$

$$\beta > k$$

if α is complex then the radius becomes a cosh curve

$$a(x) = \sqrt{\frac{A_0}{\pi}} \cos\left(i\frac{\beta x}{2}\right) = \sqrt{\frac{A_0}{\pi}} \cosh\left(\frac{\beta x}{2}\right)$$

$$A(x) = A_0 \cosh^2\left(\frac{\beta x}{2}\right)$$


a(x)
$\sqrt{\frac{A_0}{\pi}}$
0
x
cosh
curve.

d) i) $\frac{1}{a}\frac{d^2a}{dx^2} = 0$

$\Rightarrow$ all plane waves propagate

ii) $\frac{1}{a}\frac{d^2a}{dx^2} = 0$

$\Rightarrow$ all plane waves propagate.

e) bugle horns.

The bell is designed so that $\frac{1}{a}\frac{d^2a}{dx^2}$ increases, meaning lower frequencies reflect deeper inside the bell. This helps the bugle be able to produce a full harmonic series despite having closed-open boundary conditions.

Derivation of Modified Webster Horn equation.
NOT IN QUESTION.

$$k^2 \hat{p} + \frac{1}{A}\frac{d}{dx}\left(A\frac{d\hat{p}}{dx}\right) = 0$$

$$\psi(x) = \hat{p}(x)\, A(x)^{1/2}$$

$$\hat{p} = \psi A^{-1/2} \qquad A = \pi a^2$$

$$= \psi \pi^{-1/2} a^{-1}$$

$$\frac{k^2\psi}{a}\cancel{\pi^{-1/2}} + \frac{1}{\cancel{\pi}a^2}\frac{d}{dx}\left(\cancel{\pi}a^2\frac{d}{dx}\left(\psi a^{-1}\cancel{\pi^{-1/2}}\right)\right) = 0$$

$$\frac{k^2}{a}\psi + \frac{1}{a^2}\frac{d}{d\eta}\left(a^2\left(\frac{1}{a}\frac{d\psi}{d\eta} - \frac{\psi}{a^2}\frac{da}{d\eta}\right)\right) = 0$$

$$\frac{k^2}{a}\psi + \frac{1}{a^2}\frac{d}{d\eta}\left(a\frac{d\psi}{d\eta} - \psi\frac{da}{d\eta}\right) = 0$$

$$\frac{k^2}{a}\psi + \frac{1}{a^2}\left(a\frac{d^2\psi}{d\eta^2} + \cancel{\frac{da}{d\eta}\frac{d\psi}{d\eta}} - \cancel{\frac{d\psi}{d\eta}\frac{da}{d\eta}} - \psi\frac{d^2a}{d\eta^2}\right) = 0$$

$$k^2\psi + \frac{d^2\psi}{d\eta^2} - \frac{\psi}{a}\frac{d^2a}{d\eta^2} = 0$$

$$\frac{d^2\psi}{d\eta^2} + \left(k^2 - \frac{1}{a}\frac{d^2a}{d\eta^2}\right) = 0$$