Version AA/3

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 4 May 2021 9 to 10.40

Module 4A15

AEROACOUSTICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A15 Aeroacoustics data sheet (6 pages)
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 The acoustic pressure on the surface of a vibrating sphere of radius a is given by

$$p' = A \cos\theta \cos(\omega t)$$

where A and ω are constant and θ is the polar angle in spherical coordinates. Assume that the mean density and speed of sound in surrounding fluids are constant and denoted by ρ_o and c_o, respectively.

(a) Find the time averaged acoustic power generated by the source. [85%]

(b) If λ is the wavelength of sound produced, explain why the amplitude of vibration A needs to be much higher to get the same power when $\lambda \gg a$ compared to when $\lambda \ll a$. [15%]

2 A 330 ml open Coke can is to be used as a Helmholtz resonator. The Helmholtz resonance frequency is given by

$$f = \frac{c_0}{2\pi}\sqrt{\frac{A}{lV}}$$

where c_0 is the speed of sound, A and l are the area and the effective length of the neck of the resonator, respectively and V is the volume of the cavity.

(a) State all the assumptions necessary to derive the above expression. [15%]

(b) By assuming that the can has a thickness of 10^{-4} m throughout, a height of 0.1 m and a circular opening of area 4×10^{-4} m^2, derive the Helmholtz resonance frequency of the can. [25%]

(c) The measured frequency is a bit lower than the value calculated in part (b). Without doing detailed calculations, explain why this would be the case. [15%]

(d) If, by using a can opener, we remove the top of the can completely, what would be the new lowest acoustic resonance frequency? [15%]

(e) A 330 ml Coke bottle has a resonance frequency of 300 Hz. Explain why the frequency is so different from the can resonance frequency. [15%]

(f) We want the Coke bottle to produce a C4 note (262 Hz) when we blow over its top. How would you accomplish this with the 330 ml bottle? [15%]

3 If $\hat{S}_n$ are the strengths of a series of monopole sources located close to the origin at $\mathbf{x}_n$, then the acoustic pressure field is given by,

$$\hat{p}(\mathbf{x}) = \sum_n \hat{S}_n f(\mathbf{x} - \mathbf{x}_n) \qquad \text{where} \qquad f(\mathbf{x}) = \frac{1}{|\mathbf{x}|} e^{-ik|\mathbf{x}|}.$$

(a) Find the multipole series expansion for the acoustic field $\hat{p}(\mathbf{x})$ about the origin. Include the first two terms of the series. [20%]

(b) Under what circumstances will the pressure field $\hat{p}(\mathbf{x})$ far away from the origin have significant directional dependence? [10%]

(c) 4 monopole sources are placed at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, with strengths $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$, and $\hat{S}_4$ respectively. What strength of dipole, placed at the origin, could be used to model these 4 sources if $\hat{S}_1 = -\hat{S}_3 = 3$ and $\hat{S}_2 = -\hat{S}_4 = 5$? How will the accuracy of the dipole approximation vary with frequency? [15%]

If instead of a discrete set of sources, we have a continuous distribution $\hat{S}(\mathbf{x})$, the acoustic field $\hat{p}(\mathbf{x})$ will be governed by the equation,

$$(\nabla^2 + k^2)\hat{p}(\mathbf{x}) = 4\pi \hat{S}(\mathbf{x}).$$

(d) Use the Green's function to find the integral expression for $\hat{p}(\mathbf{x})$. [10%]

(e) By doing a Taylor series expansion of the Green's function derive the effective monopole and dipole strengths at the origin that can be used to approximate the sound field produced by $\hat{S}(\mathbf{x})$. [20%]

A sphere of radius a, oscillates along the x_1 axis with velocity amplitude U and frequency ω. $k = \omega/c_0$ is the wavenumber, and (r, θ, ψ) are spherical polar coordinates. If the sphere is compact the acoustic field is given by,

$$\hat{p}(r,\theta) = -\frac{1}{2} i\omega\rho_0 U a^3 \cos\theta \frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right).$$

(f) What are the effective monopole and dipole strengths of the oscillating sphere? [25%]

4 (a) Plane waves travel in a duct, aligned with the x-axis, that has varying cross-sectional area $A(x)$. Show that these waves are governed by the Webster horn equation

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{A}\frac{\partial}{\partial x}\left(A\frac{\partial p'}{\partial x}\right) = 0.$$

[20%]

If we make the substitutions $\psi(x) = \hat{p}(x)\sqrt{A(x)}$ and $A(x) = \pi a(x)^2$, where $a(x)$ is the local radius of the duct, then the Webster horn equation can be written

$$\frac{d^2\psi}{dx^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{dx^2}\right)\psi = 0. \qquad (1)$$

where $k = \omega/c_0$ is the wavenumber.

(b) $A(x)$ takes the form $A(x) = A_0[1+\cos(2\alpha x)]/2$. A_0 is a real constant and α is a complex constant. What form does the pressure field take? [35%]

(c) What value of α could cause waves not to propagate in the duct? Sketch the corresponding shape of the duct. [20%]

(d) What does equation (1) tell us about wave propagation for:

(i) a cylindrical duct?

(ii) a conical duct?

[15%]

(e) Using an example of a musical instrument explain why it is beneficial to have a duct in which not all plane waves of all frequencies can propagate. [10%]

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Module 4A15 Aeroacoutics Data Sheet

USEFUL DATA AND DEFINITIONS

Physical Properties

Speed of sound in an ideal gas $\sqrt{\gamma RT}$, where T is temperature in Kelvins

Units of sound measurement

$$\begin{aligned}
\text{SPL (sound pressure level)} &= 20\log_{10}\left(\frac{p'_{rms}}{2\times10^{-5}\text{Nm}^{-2}}\right)\text{dB} \\
\text{IL (intensity level)} &= 10\log_{10}\left(\frac{\text{intensity}}{10^{-12}\text{watts m}^{-2}}\right)\text{dB} \\
\text{PWL (power level)} &= 10\log_{10}\left(\frac{\text{sound power output}}{10^{-12}\text{watts}}\right)\text{dB}
\end{aligned}$$

Definitions

Surface impedance Z_s, relates the pressure perturbation applied to a surface, p', to its normal velocity v'; $p' = Z_s v'$

Characteristic impedance of a fluid $\rho_0 c_0$

Specific impedance of a surface $Z_s/(\rho_0 c_0)$

Wavenumber $k = \omega/c_0 = 2\pi/\lambda$, where λ is the wavelength

Helmholtz number (or compactness ratio) = kD, where D is a typical dimension of the source.

Strouhal number $= \omega D/(2\pi U)$ for sound of frequency ω (in rad/s), produced in a flow with speed U, length scale D.

Basic equations for linear acoustics

Conservation of mass

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0$$

Conservation of momentum

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0$$

Isentropic

$$c_0^2 = \left. \frac{dp}{d\rho} \right|_S$$

Wave equation

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

Energy density

$$e = \frac{1}{2} \rho_0 v'^2 + \frac{1}{2\rho_0 c_0^2} p'^2$$

Intensity $\mathbf{I} = p'\mathbf{v}'$

Velocity potential ϕ' satisfies the wave equation and $p' = -\rho_0 \frac{\partial \phi'}{\partial t}$, $\mathbf{v}' = \nabla \phi'$.

Autocorrelation $F(\xi)$, the autocorrelation of $f(y)$ is given by

$$F(\xi) = \overline{f(y) f(y+\xi)}$$

$$F(0) = \overline{f^2}$$

Integral length scale, l

$$l\overline{f^2} = \int_{-\infty}^{\infty} F(\xi) d\xi$$

Sound power

Sound power from a source is defined as

$$P = \int_S \bar{\mathbf{I}} \cdot \mathbf{dS} = \int_{S_\infty} \frac{\overline{p'^2}}{\rho_0 c_0} \mathbf{dS}$$

for a statistically stationary source. For an outward propagating spherically symmetrical sound field $P = \frac{\overline{p'^2}}{\rho_0 c_0} 4\pi r^2$, where p' is the acosutic pressure at radius r.

For a sound field, which is a function of spherical polar coordinates r, θ only, and is independent of the azimuthal angle,

$$P = 2\pi r^2 \int_0^{\pi} \frac{\overline{p'^2}}{\rho_0 c_0} \sin\theta d\theta$$

Simple wave fields

1D or plane wave

The general solution of the 1D wave equation is $p'(x,t) = f(t - x/c_0) + g(t + x/c_0)$, where f and g are arbitrary functions. In a plane wave propagating to the right $p' = \rho_0 c_0 u'$; in a plane wave propagating to the left $p' = -\rho_0 c_0 u'$, u' being the particle velocity.

Spherically symmetric sound fields

The general spherically symmetric solution of the 3D wave equation is

$$\phi'(r,t) = \frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r},$$

where r is the distance from the source; f and g are arbitrary functions.

$\cos\theta$ dependence

The general solution of the 3D wave equation with $\cos\theta$ dependence is

$$p'(\mathbf{x},t) = \frac{\partial}{\partial x}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right] = \cos\theta \frac{\partial}{\partial r}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right]$$

Useful mathematical formulae

Spherical polar coordinates (r, θ, ψ)

Gradient

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p'}{\partial \psi}\right)$$

Divergence

$$\nabla \cdot \mathbf{v}' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r'\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta v_\theta'\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi'}{\partial \psi}$$

Laplacian

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p'}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial \psi^2}$$

Delta functions

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1D δ-function $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(ax-b) f(x) dx = f(b/a)/|a|$

3D δ-function $\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3)$

Convolution algebra

Convolution of $f(\mathbf{x})$ and $g(\mathbf{x})$

$$(f \star g)(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

Commutative properties

$$f \star g = g \star f$$

$$\frac{\partial}{\partial x_i}(f \star g)(\mathbf{x}) = f \star \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \star g$$

Green's function

3D Green's function for wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 \right) g(\mathbf{x}, t | \mathbf{y}, \tau) = \delta(t - \tau) \delta(\mathbf{x} - \mathbf{y})$$

$$g(\mathbf{x}, t | \mathbf{y}, \tau) = \frac{\delta \left\{ |\mathbf{x} - \mathbf{y}| - c_0 (t - \tau) \right\}}{4\pi c_0 |\mathbf{x} - \mathbf{y}|}$$

Lighthill's Acoustic Analogy

Lighthill's equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 \right) \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

For cold, isentropic, low Mach-number jets, T_{ij} can be approximated as:

$$T_{ij} = \rho_0 u_i u_j$$

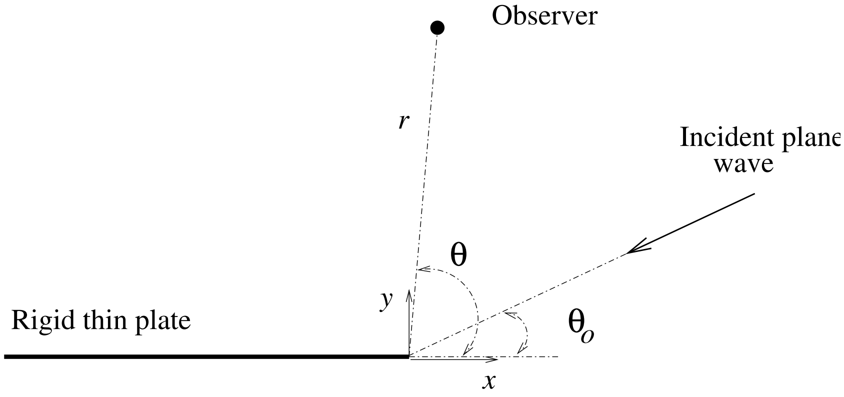

Figure 1: Geometry for edge scattering

Lighthill eight power law Acoustic power,

$$P_a \sim \frac{\rho_o d_j^2}{c_0^5} u_j^8,$$

where d_j and u_j are the jet exit diameter and velocity, respectively.

Refraction

Snells's law for determining a ray path is

$$\frac{\sin\theta}{c_0} = \text{constant} \ . \tag{1}$$

Diffraction

Field scattered by a sharp edge If the incident plane waves is

$$p_\mathrm{i}(\mathbf{x},t) = P_\mathrm{inc} \exp(\mathrm{i}\omega t + \mathrm{i}k_0 x \cos\theta_0 + \mathrm{i}k_0 y \sin\theta_0) \ , \tag{2}$$

then the diffracted pressure is

$$p_\mathrm{d} = P_\mathrm{inc} \left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} \frac{\sin(\theta_0/2)\sin(\theta/2)}{\cos\theta + \cos\theta_0} \exp\left(-\frac{\mathrm{i}\pi}{4} - \mathrm{i}k_0 r\right) \ . \tag{3}$$

In a cylindrical duct of radius a

The pressure field is given by

$$p'(\mathbf{x},t) = e^{i(\omega t + n\theta)} J_n(z_{mn} r/a)(Ae^{-ikx_3} + Be^{ikx_3}),$$

where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_0^2 - z_{mn}^2/a^2)^{1/2}$.

For large azimuthal wavenumber, n

$$z_{1n} \approx n + 1.85n^{1/3}$$