Engineering Part IIB 2008

4A8 – Environmental Fluid Mechanics

Numerical Answers

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Q1: -

Q2: (b) 9.51 m; 0.126 m/s; 0.019 N/m^2; −18 W/m^2
(c) 3.47×10^{-3} m^2/s^3 ; -5.26×10^{-4} m^2/s^3.

Q3: -

Q4: (b) 60 s.

2008 IIB 4A8 ENVIRONMENTAL FLUID MECHANICS
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1) a) The flows created when the pressure

i) forces are balanced by Coriolis forces, are called Geostrophic flow. They are governed by

$$\frac{1}{\rho} \nabla P = -2 \Omega \times \underline{u}$$

where P includes the centrifugal forces also

The boundary layer between a geostrophic flow and a solid boundary with no-slip condition is called Ekman layer.
Within the Ekman layer, the velocity vector changes its direction slowly to align with the geostrophic velocity at the end of the layer. The tip of the velocity vectors describes a spiral in space, which is called Ekman spiral.

In the Ekman layer, the balance is between Coriolis & viscous forces.

ii) Brunt-Vaisala frequency is the natural frequency of oscillation (or resonant frequency), which occurs when a fluid parcel is displaced in a stratified flow with stable stratification.

(iii) Features of turbulence:

Unsteady, Three-dimensional in nature, random & irregular, involves a wide range of scales, highly diffusive and dissipative, and rotational. It obeys the laws of continuum.

Closure Problem:

In the statistical methods of analysis of turbulence, governing equations for mean quantities involve correlations of fluctuating quantities. These correlations are unknown.

If governing equations for the correlations are derived, it will involve high order correlations which are also unknown. The appearance of correlations cannot be avoided unless modelling approximations are used. This is called turbulence closure problem.

(iv) TKE - in the data sheet.

Sources: 1) mean wind shear
2) Bouyancy - unstable temperature gradient.
3) Topography - wakes

Sinks: 1) viscous dissipation
2) Buoyancy - Stable Temperature (or) density stratification.

b) $Pv = RT$, hydrostatics: $\frac{dp}{dz} = -\rho g$

I & IInd laws of Thermodynamics.

$$Tds = dh - vdp$$

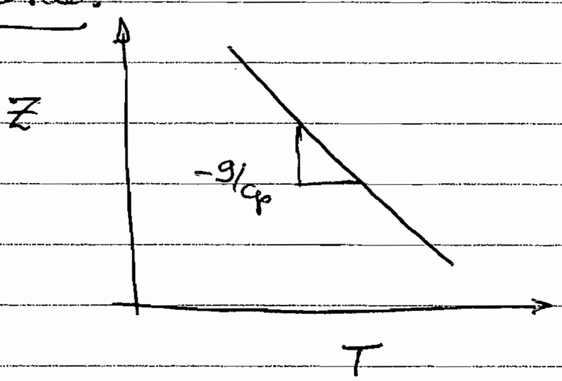
reversible adiabatic process $\Rightarrow$ $ds = 0$

$$dh = +v\,dp \Rightarrow \frac{dp}{dz} = +\frac{c_p}{v}\frac{dT}{dz}$$

$$-\rho g = +\rho c_p \frac{dT}{dz}$$

$$\Rightarrow \boxed{\frac{dT}{dz} = -g/c_p.}$$

Significance:



As $z\uparrow$ $P\downarrow \Rightarrow \rho\downarrow$
But $T\downarrow$ as $z\uparrow$
$\Rightarrow \rho\uparrow$

$\Rightarrow$ Drop in ρ because of decrease in P is compensated by the decrease in T.

$\Rightarrow$ a fluid parcel displaced by Δz, stays there.
"Neutrally stable"

2)

$$\frac{dU}{dz} = \frac{u_*}{kz}\left(1 + 4.7\, z/L\right) \qquad k = 0.4$$

a) $$\int_0^U dU = \frac{u_*}{k}\int_{z_0}^{z}\left(\frac{1}{z} + \frac{4.7}{L}\right)dz$$

$$U - U_0 = \frac{u_*}{k}\left[\ln\frac{z}{z_0} + \frac{4.7}{L}(z - z_0)\right]$$

$U_0 = 0$

$$\Rightarrow \boxed{U = \frac{u_*}{k}\left[\ln\left(\frac{z}{z_0}\right) + \frac{4.7}{L}(z - z_0)\right]}$$

b)

$$2 = \frac{u_*}{k}\left[\ln 50 + \frac{4.7}{L} * 4.9\right] \quad — ①$$

$$3 = \frac{u_*}{k}\left[\ln 100 + 4.7 * \frac{9.9}{L}\right] \quad — ②$$

$$\frac{②}{①} \Rightarrow 1.5 = \frac{4.61L + 46.53}{3.91L + 23.03} \Rightarrow \boxed{L = 9.51\,m}$$

from ②

$$u_* = 3 * 0.4\left[\ln 100 + \frac{46.53}{L}\right]^{-1}$$

$$\Rightarrow \boxed{u_* = 0.126 \ m/sec}$$

Shear Stress at the surface $\tau_w = \rho u_*^2$
(in the datasheet)

$$\tau_w = 1.2 * (0.126)^2 = 0.0191 \ N/m^2$$

$$\boxed{\tau_w = 0.0191 \ N/m^2}$$

Surface heat flux (Q)

from data sheet $L = -\dfrac{u_*^3}{k \, g/T \, Q/\rho C_p}$

$$\Rightarrow Q = -\frac{u_*^3 \rho C_p T}{k g L} = -0.018 \ kW/m^2$$

$$\boxed{Q = -18 \ W/m^2}$$

c) TKE production via mean shear

$-\overline{u_i' u_k'} \dfrac{\partial \bar{U}_i}{\partial x_k}$ from data sheet.

$$\simeq u_*^2 \left(\frac{d\bar{U}}{dz}\right)_{z=5}$$

$$= (0.126)^2 * \frac{0.126}{0.4*5}\left(1 + 4.7\frac{5}{9.51}\right) = 3.47 \times 10^{-3}$$

Production via mean shear $= \underline{\underline{3.47 \times 10^{-3} \ m^2/s^3}}$

via Buoyancy mechanism:

$$\frac{-gQ}{\rho C_p T} \quad \text{(from the data sheet)}$$

$$= \frac{9.81 * 18\times10^{-3}}{1.2\times1\times280} = -5.26\times10^{-4} \quad m^2/s^3.$$

d) $$\frac{\text{Thermal}}{\text{mechanical}} = \frac{-gQ/\rho C_p T}{u_*^2\left(\frac{d\bar{u}}{dz}\right)}$$

$$u_*^2\left(\frac{d\bar{u}}{dz}\right) = \frac{u_*^3}{\cancel{k}\cancel{z}}\,\frac{4.7\cancel{z}}{L} \quad \text{But} \quad L \simeq \frac{-u_*^3\rho C_p T}{k g Q}$$

$$= -\frac{\cancel{u_*^3}}{\cancel{k}}\,\frac{4.7\,k g Q}{\cancel{u_*^3}\rho C_p T} = -4.7\,\frac{gQ}{\rho C_p T}$$

$$\Rightarrow \frac{\text{thermal dissipation}}{\text{mechanical production}} = \frac{1}{4.7} = \underline{\underline{0.213}}$$

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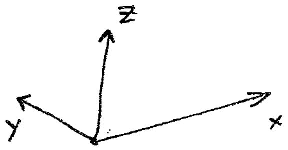

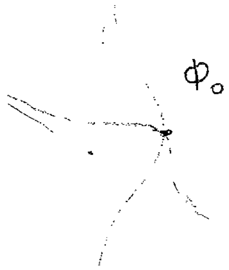

At $y = z = 0$, $\quad \bar{\Phi} = \phi_0 = \dfrac{Q}{2\pi}\dfrac{1}{U}\dfrac{1}{\sigma^2}\exp\left[\dfrac{0}{2\sigma^2} + \dfrac{0}{2\sigma^2}\right]$

$$\Longrightarrow \phi_0 = \frac{Q}{2\pi}\frac{1}{U}\frac{1}{\sigma^2} \qquad (\text{equal } \sigma)$$

(a) Close to the surface, we expect the vertical lengthscale to be limited. The strong shear of the velocity profile also implies turbulence generation of the streamwise component of the velocity fluctuations.

Hence we expect $\sigma_y > \sigma_z$ for neutral stability.

(For strongly unstable situations, buoyancy production in the vertical direction makes $\sigma_z > \sigma_y$ in general)

(b) Transport equation for the fluctuations includes production term $\left[2K\left(\dfrac{\partial\bar{\Phi}}{\partial x_i}\right)^2\right]$, which by neglecting gradients in x-direction is $2K\left[\left(\dfrac{\partial\bar{\Phi}}{\partial y}\right)^2 + \left(\dfrac{\partial\bar{\Phi}}{\partial z}\right)^2\right]$

Dissipation term is $\dfrac{2}{T_{turb}}\,g$.

From Gaussian plume equation

$$\bar{\Phi} = \frac{Q}{2\pi}\frac{1}{U}\frac{1}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right)\cdot\exp\left(-\frac{z^2}{2\sigma^2}\right)$$

we get

$$\frac{\partial \bar{\Phi}}{\partial y} = \overbrace{\frac{Q}{2\pi}\frac{1}{U}\frac{1}{\sigma^2}}^{\Phi_0}\left(-\frac{2y}{2\sigma^2}\right)\exp\left(-\frac{y^2}{2\sigma^2}\right)\exp\left(-\frac{z^2}{2\sigma^2}\right)$$

$$\Leftrightarrow \left(\frac{\partial \bar{\Phi}}{\partial y}\right)^2_{y=z=\sigma} = \Phi_0^2 \frac{\sigma^2}{\sigma^4} \exp(-1)\exp(-1) = \Phi_0^2 \frac{1}{\sigma^2} e^{-2}$$

and similarly for $\left(\frac{\partial \bar{\Phi}}{\partial z}\right)^2_{y=z=\sigma}$

$$\Rightarrow \text{Production term} = 2K \cdot \frac{\Phi_0^2}{\sigma^2} e^{-2} + 2K \frac{\Phi_0^2}{\sigma^2} e^{-2}$$

$$= 4 \frac{\Phi_0^2 K}{\sigma^2} e^{-2}$$

Dissipation = Production implies

$$\frac{2g}{T_{turb}} = 4 \frac{\Phi_0^2 K}{\sigma^2} e^{-2}$$

$$\Leftrightarrow \frac{g}{\Phi_0^2} = \frac{2 K T_{turb}}{\sigma^2} e^{-2}$$

$$\Rightarrow \frac{g^{1/2}}{\Phi_0} = \frac{(2 K T_{turb})^{1/2}}{\sigma} e^{-1} \qquad \text{QED.}$$

(E) The correct turbulent lengthscale to use here $(\sigma \sim x^1)$ is σ. Hence $K = C \cdot u'\sigma$ and since $T_{turb} = \frac{\sigma}{u'}$,

$$\frac{g^{1/2}}{\Phi_0} = \frac{\left(2 C \cdot u'\sigma \frac{\sigma}{u'}\right)^{1/2}}{\sigma} e^{-1}$$

$\Rightarrow$ for $\frac{g^{1/2}}{\Phi_0} = 0.25$, $C = 0.23$

$$\Rightarrow \boxed{K = 0.23\, u'\sigma}$$

4. (a) Mass balance for pollutant in street canyon.

$$\begin{pmatrix}\text{rate of}\\ \text{accumulation}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{emission}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{efflux from top}\end{pmatrix}$$

$$\frac{d}{dt}\left(c \cdot HWL\right) = Q \cdot WL - \dot{m}' \cdot WL$$

$$H \frac{dc}{dt} = Q - u(c-b)$$

$$\Leftrightarrow \boxed{\frac{dc}{dt} = \frac{Q}{H} - \frac{u}{H}(c-b)} \qquad \text{QED}$$

(b) At long times (steady-state), $\frac{dc}{dt} = 0$

$$\Rightarrow c = b + \frac{Q}{u} \quad \Rightarrow c = 2 \times 10^{-8} \text{ kg/m}^3$$

(note: this is not needed)

At $t=0$, $Q=0$

$$\Rightarrow \frac{dc}{dt} = -\frac{u}{H} c$$

$$\Rightarrow c = c_0 \exp\left(-\frac{u}{H} t\right)$$

For $\frac{c}{c_0} = 0.05$, $\frac{u}{H} t \simeq 3 \Rightarrow t = \frac{3H}{u}$

$$\Rightarrow \boxed{t = 60 \text{ s}}$$

(C). The exchange velocity u may be affected by:

- wind speed U (high U, implies high turbulent intensity $\Rightarrow$ high u)
- hot street ($\Rightarrow$ buoyant production of turbulence)
- Upwind street canyons or obstacles, which may affect local velocity profile $\Rightarrow$ shear
- Time of day (night or day), sunlight levels
- Stability of atmosphere