Worked solutions for 4B23 Optical Fibre Communication 2022/23 paper version 3

Question 1

a) From the 4B23 formula sheet

$$n_g = n - \lambda \frac{dn}{d\lambda}$$

Therefore

$$n_g = A + \left(\frac{B}{\lambda}\right)^2 - \left(\frac{\lambda}{b}\right)^2 - \lambda\left[-\frac{2B^2}{\lambda^3} - \frac{2\lambda}{b^2}\right] = A + 3\left(\frac{B}{\lambda}\right)^2 + \left(\frac{\lambda}{b}\right)^2$$

Also from the 4B23 formula sheet

$$D = -\frac{\lambda}{c}\frac{d^2n}{d\lambda^2} = -\frac{\lambda}{c}\frac{d}{d\lambda}\left[-\frac{2B^2}{\lambda^3} - \frac{2\lambda}{b^2}\right] = -\frac{\lambda}{c}\left[\frac{6B^2}{\lambda^4} - \frac{2}{b^2}\right] = -\frac{6B^2}{c}\frac{1}{\lambda^3} + \frac{2\lambda}{cb^2}$$

We know at $\lambda = 1312$ nm $D = 0$ therefore

$$\lambda_0^4 = 3B^2b^2$$

Hence we can re-write as

$$D = \frac{2}{cb^2}\left(\lambda - \frac{\lambda_0^4}{\lambda^3}\right)$$

Therefore rearranging gives

$$b = \sqrt{\frac{2\lambda}{cD}\left(1 - \frac{\lambda_0^4}{\lambda^4}\right)}$$

We note that

$$c = 3 \times 10^8 \frac{m}{s} = 3 \times 10^8 \frac{m}{s} \times 10^{-3} \frac{km}{m} \times 10^{-12} \frac{s}{ps} = 3 \times 10^{-7} \frac{km}{ps}$$

Using c in km/ps allows the use of D in ps/nm/km with wavelengths λ and λ_0 in nm such that

$$b = \sqrt{\frac{2 \times 1550}{3 \times 10^{-7} \times 17}\left(1 - \frac{1312^4}{1550^4}\right)} = 17199 \text{ nm}$$

Therefore from $\lambda_0^4 = 3B^2b^2$

$$B = \frac{1312^2}{17199 \times \sqrt{3}} = 57.78 \text{ nm}$$

Therefore from $n_g = A + 3\left(\frac{B}{\lambda}\right)^2 + \left(\frac{\lambda}{b}\right)^2$ we have

$$A = 1.4620 - 3\left(\frac{57.78}{1550}\right)^2 - \left(\frac{1550}{17199}\right)^2 = 1.4497$$

30 marks

b) Minimum latency for the data will happen when group refractive index is a minimum. The overall minimum occurs at 1312 nm so the first wavelength channel at 1530 nm will experience the minimum latency

The group refractive index is $1.4497 + 3 \times \left(\frac{57.78}{1530}\right)^2 + \left(\frac{1530}{17199}\right)^2 = 1.4619$

Hence the latency in s will be $1.4619 \times \frac{6500\times10^3}{3\times10^8} = 0.03167 = 31.7\ \mathrm{ms}$
Alternative method
Latency at 1550 nm is $1.4620 \times \frac{6500\times10^3}{3\times10^8} = 0.03168 = 31.7\ \mathrm{ms}$

Dispersion is 17 nm at 1550 nm so difference at 1530 nm is $20\ \times 6500 \times\ 17 = 2\ \mu\mathrm{s}$
So at 1530 nm latency is 31.7 ms

10 marks

c) Rectangular Nyquist spectrum means that a 100.5 GBd signal will occupy 100.5 GHz of optical spectrum. Maximum dispersion will occur at 1610 nm

$$D = \frac{2}{cb^2}\left(\lambda - \frac{\lambda_0^4}{\lambda^3}\right)$$

With $b = 17199\ \mathrm{nm}$ and $\lambda_0 = 1312\ \mathrm{nm}$ hence

$$D = \frac{2 \times 1610}{3 \times 10^{-7} \times (17199)^2}\left(1 - \left[\frac{1310}{1610}\right]^4\right) = 20.4\ \mathrm{ps/nm/km}$$

At $\lambda = 1610\ \mathrm{nm}$ the frequency is $f = 3 \times 10^8/1.61 \times 10^{-6} = 1.86 \times 10^{14}\ \mathrm{Hz}$

$$\Delta\lambda = \lambda \times \frac{\Delta f}{f} = 1610 \times \frac{100.5 \times 10^9}{1.86 \times 10^{14}} = 0.87\ \mathrm{nm}$$

Hence the delay from 6500 km is

$$6500 \times 20.4 \times 0.87 = 115.3\ \mathrm{ns}$$

Sampling rate is $100.5 \times \frac{16}{15} = 107.2$ GSa/s

Therefore number of samples corresponding to 115.3 ns is

$$N_{CD} = 115.3 \times 10^{-9} \times 107.2 \times 10^9 = 12360.1$$

d) Using the overlap and save algorithm with an N point FFT the number of complex multiplies per sample N_{cm} is

$$N_{cm} = \frac{N\log_2(N) + N}{N - N_{CD} + 1}$$

Given $N_{CD} = 12360$ we expect the minimum value of N to be $N = 2^{15} = 32768$ which gives

$$N_{cm} = \frac{32768 \times 15 + 32768}{32768 - 12360 + 1} = 25.7$$

Given there are no technological limitations regarding the FFT size let us consider $N = 2^{16}$ which gives

$$N_{cm} = \frac{2^{16} \times 17}{2^{16} - 12360 + 1} = 20.95$$

Increasing to $N = 2^{17}$ gives

$$N_{cm} = \frac{2^{17} \times 18}{2^{17} - 12360 + 1} = 19.87$$

Increasing to $N = 2^{18}$ gives

$$N_{cm} = \frac{2^{18} \times 19}{2^{18} - 12360 + 1} = 19.94$$

Hence the optimum value of $N = 2^{17}$. The power consumption per polarisation is

$P = 19.87 \times 0.5 \times 10^{-12} \times 107.2 \times 10^{9} = 1.07\ W$ and hence for two polarisations the total power consumption is 2.1 W.

30 marks

Question 2

a)

$$\frac{dn(z)}{dz} = \eta(N_2 - N_1)n(z) + \eta N_2$$
$$\frac{d}{dz}\{n(z)e^{-\eta(N_2-N_1)z}\} = \eta N_2 e^{-\eta(N_2-N_1)z}$$

$$n(z)e^{-\eta(N_2-N_1)z} = -\frac{N_2}{N_2 - N_1}e^{-\eta(N_2-N_1)z} + C$$

For $z = 0$ we have $n(0) = n_{in}$ and hence

$$n_{in} = -\frac{N_2}{N_2 - N_1} + C$$

Therefore

$$n(z)e^{-\eta(N_2-N_1)z} = -\frac{N_2}{N_2 - N_1}e^{-\eta(N_2-N_1)z} + \left(n_{in} + \frac{N_2}{N_2 - N_1}\right)$$

Therefore, for a length of erbium fibre of length L defining $G = e^{\eta(N_2-N_1)L}$ to be the gain and $n_{sp} = N_2/(N_2 - N_1)$ the number of photons exiting will be

$$\frac{n(L)}{G} = -\frac{n_{sp}}{G} + (n_{in} + n_{sp})$$

therefore

$$n(L) = n_{in}G + n_{sp}(G - 1)$$

20 marks

b)

i) $n_{sp} = \frac{N_2}{N_2-N_1}$ with $N_2 = 2N_1$ gives $n_{sp} = \frac{N_2}{N_2-N_1} = \frac{2N_1}{2N_1-N_1} = 2$

The power spectral density of the ASE is $N_{ASE} = 2n_{sp}h\nu(G - 1) = 4h\nu(G - 1)$

The noise in 64 GHz is $4 \times 1.3 \times 10^{-19} \times 64 \times 10^9(G - 1) = 33.28 \times 10^{-9}(G - 1)$

With span of 80 km $G = 10^{80\times\frac{0.2}{10}} = 39.8$

PSD of ASE is $4 \times 1.3 \times 10^{-19} \times 38.8 = 2.02 \times 10^{-5} pJ$

noise from ASE is 1.291$\times 10^{-6}$=1.29 μW

So SNR is 1000/1.29=774.6 which is 28.8 dB
After 480 km corresponding to 6 spans the SNR is 129.1 which is 21.1 dB

20 marks

ii) $\alpha_{dB} = 0.2$ dB/km (so $\alpha = 0.046$ km^{-1}), D=17 ps/nm/km (so $|\beta_2| = 17/0.784 = 21.6$ ps^2/km), $B_o = 5$ THz, $\gamma = 1.3$ W^{-1}km^{-1}, $L = 100$ km (so $L_{eff} = 23.8$ km).

$$C_{NLI} = \frac{8\gamma^2 L_{eff}^2 \alpha}{27\pi|\beta_2|}\ln\left(\frac{\pi^2|\beta_2|B_o^2}{\alpha}\right) = (1 - e^{-\alpha L})^2 \frac{8\gamma^2}{27\pi|\beta_2|\alpha}\ln\left(\frac{\pi^2|\beta_2|B_o^2}{\alpha}\right)$$

$$C_{NLI} = (1-e^{-\alpha L})^2 \frac{8\times 1.3^2}{27\pi|21.6|0.046} \ln\left(\frac{\pi^2|21.6|5^2}{0.046}\right)$$
$$= (1-e^{-\alpha L})^2 \times 0.16 \times 11.66 \ (pJ)^{-2} = (1-e^{-\alpha L})^2 \times 1.87 \ (pJ)^{-2}$$

With 80 km PSD is 1.78 $(pJ)^{-2}$

So optimum signal PSD is $\sqrt[3]{\frac{2.02\times10^{-5}}{2\times1.78}} = 0.0178\, pJ$

So for a 64 GBd signal $64\times10^{-3}\times0.0178 = 1.1\ mW$

Therefore

$$PSD_{opt} = \sqrt[3]{\frac{5.2\times10^{-7}\times[\exp(\alpha\, L_s)-1]}{2\times1.87[[1-\exp(-\alpha\, L_s)]^2}} = \sqrt[3]{\frac{5.2\times10^{-7}\times[\exp(\alpha\, L_s)-1]}{2\times1.87[1-\exp(-\alpha\, L_s)]^2}}$$
$$= 5.18\times10^{-3}\sqrt[3]{\frac{[\exp(\alpha\, L_s)-1]}{[1-\exp(-\alpha\, L_s)]^2}} = 5.18\times10^{-3}\sqrt[3]{\frac{\exp\,(2\alpha L_s)}{\exp(\alpha\, L_s)-1}}$$

And $SNR_{opt} = \frac{2PSD_{opt}}{3\times4\times1.3\times10^{-7}(\exp(\alpha L_s)-1)} = \frac{\frac{2\times5.18\times10^{-3}}{3\times4\times1.3\times10^{-7}}\exp\left(\frac{2}{3}\alpha L_s\right)}{[\exp(\alpha L_s)-1]^{4/3}} = \frac{6641}{[\exp(\alpha L_s/2)-\exp(-\alpha L_s/2)]^{\frac{4}{3}}} = \frac{2635}{\left[\sinh\left(\frac{\alpha L_s}{2}\right)\right]^{\frac{4}{3}}}$

For $L = 80$ km

$$SNR_{opt} = 590.85$$

Hence after 6 spans SNR_{opt} =98 giving 19.9 dB

20 marks

iii)

For initial design with span of 60 km $G = 10^{60\times\frac{0.2}{10}} = 15.85$

Therefore noise from ASE is 0.494×10^{-6}

So SNR is 2024 which is 33.1 dB

After 480 km corresponding to 8 spans the SNR is 253 which is 24 dB so SNR increases by 3 dB

For the more detailed design

$$SNR_{opt} = \frac{2635}{\left[\sinh\left(\frac{\alpha L_s}{2}\right)\right]^{\frac{4}{3}}}$$

For $L = 60\ km$

$$SNR_{opt} = 1150.6$$

Hence after 8 spans $SNR_{opt} = 143.8$ giving 21.6 dB

So increase of 1.7 dB (with reduction do to NL effects)

c) The power per EDFA is modelled as $kG = k \exp{(\alpha L_s)}$

$$P_T = \frac{kL}{L_s}\exp(\alpha L_s)$$

$$\frac{dP_T}{dL_s} = -\alpha \frac{kL}{L_s}\exp(\alpha L_s) - \frac{kL}{L_s^2}\exp(\alpha L_s) = 0$$

Therefore $\frac{\alpha}{L_s} = \frac{1}{L_s^2}$ therefore $L_s = \frac{1}{\alpha}$

20 marks

Question 3

a) $V = \frac{2\pi a}{\lambda}\sqrt{n_{co}^2 - n_{cl}^2}$

At cutoff wavelength $V = 2.405$ so

$$\lambda_c = \frac{2\pi a}{2.405}\sqrt{n_{co}^2 - n_{cl}^2} = \frac{2\pi \times 5 \times 10^{-6}}{2.405}\sqrt{1.45^2 - 1.446^2} = 1406\ \text{nm}$$

10 marks

b)

i) With $E(r) = E_0 \exp\left(-\frac{r^2}{r_0^2}\right)$

It follows that

$$\frac{dE}{dr} = -\frac{2r}{r_0^2}E_0 \exp\left(-\frac{r^2}{r_0^2}\right)$$

And hence substituting into the expression for β gives

$$\beta^2 = \frac{k_0^2 \int_0^\infty rn^2 \exp\left(-2\frac{r^2}{r_0^2}\right) dr - \int_0^\infty 4\frac{r^3}{r_0^4}\exp\left(-2\frac{r^2}{r_0^2}\right) dr}{\int_0^\infty r \exp\left(-2\frac{r^2}{r_0^2}\right) dr}$$

We note that

$$\int_0^\infty r\exp\left(-2\frac{r^2}{r_0^2}\right) dr = -\frac{r_0^2}{4}\int_0^\infty -4\frac{r}{r_0^2}\exp\left(-2\frac{r^2}{r_0^2}\right) dr = -\frac{r_0^2}{4}\left[\exp\left(-2\frac{r^2}{r_0^2}\right)\right]_0^\infty$$
$$= \frac{r_0^2}{4}$$

$$\int_0^\infty rn^2\exp\left(-2\frac{r^2}{r_0^2}\right) dr = \int_0^a rn_{co}^2\exp\left(-2\frac{r^2}{r_0^2}\right) dr + \int_a^\infty rn_{cl}^2\exp\left(-2\frac{r^2}{r_0^2}\right) dr$$
$$= n_{co}^2\left[-\frac{r_0^2}{4}\exp\left(-2\frac{r^2}{r_0^2}\right)\right]_0^a + n_{cl}^2\left[-\frac{r_0^2}{4}\exp\left(-2\frac{r^2}{r_0^2}\right)\right]_a^\infty$$
$$= \frac{n_{co}^2 r_0^2}{4}\left[1 - \exp\left(-2\frac{a^2}{r_0^2}\right)\right] + \frac{n_{cl}^2 r_0^2}{4}\exp\left(-2\frac{a^2}{r_0^2}\right)$$

- And $\int_0^\infty 4\frac{r^3}{r_0^4}\exp\left(-2\frac{r^2}{r_0^2}\right) dr = -\int_0^\infty \frac{r^2}{r_0^2}\left(-4\frac{r}{r_0^2}\right)\exp\left(-2\frac{r^2}{r_0^2}\right) dr =$
$-\left[\frac{r^2}{r_0^2}\exp\left(-2\frac{r^2}{r_0^2}\right)\right]_0^\infty + \int_0^\infty \frac{2r}{r_0^2}\exp\left(-2\frac{r^2}{r_0^2}\right) dr = \frac{1}{2}\left[-\exp\left(-2\frac{r^2}{r_0^2}\right)\right]_0^\infty = \frac{1}{2}$

Therefore for a step index fibre the integral can be evaluated to give

$$\beta_0^2 = \frac{\frac{1}{4}\left(1 - \exp\left(-2\frac{a^2}{r_0^2}\right)\right) r_0^2 k_0^2 n_{co}^2 + \frac{1}{4}\exp\left(-2\frac{a^2}{r_0^2}\right) r_0^2 k_0^2 n_{cl}^2 - \frac{1}{2}}{\frac{r_0^2}{4}}$$

which on simplifying becomes

$$\beta^2 = \left(1 - \exp\left(-2\frac{a^2}{r_0^2}\right)\right) n_{co}^2 k_0^2 + \exp\left(-2\frac{a^2}{r_0^2}\right) n_{cl}^2 k_0^2 - \frac{2}{r_0^2}$$

20 marks

ii) To determine the optimal value of r_0 we differentiate β_0^2 with respect to r_0 to give

$$\frac{d\beta^2}{dr_0} = -4\frac{a^2}{r_0^3}\exp\left(-2\frac{a^2}{r_0^2}\right) n_{co}^2 k_0^2 + 4\frac{a^2}{r_0^3}\exp\left(-2\frac{a^2}{r_0^2}\right) n_{cl}^2 k_0^2 + \frac{4}{r_0^3} = 0$$

hence

$$1 = k_0^2 a^2 (n_{co}^2 - n_{cl}^2)\exp\left(-2\frac{a^2}{r_0^2}\right) = V^2 \exp\left(-2\frac{a^2}{r_0^2}\right)$$

Therefore

$$V = \exp\left(\frac{a^2}{r_0^2}\right)$$

$$r_0 = \frac{a}{\sqrt{\ln V}}$$

10 marks

c) mean power density is given by $\langle S \rangle$, the time averaged Poynting Vector

$$\langle S \rangle = \frac{1}{2}\frac{|E|^2}{\eta} = \frac{1}{2}\frac{n}{\eta_0}|E|^2 = \frac{1}{2}\frac{n}{\eta_0}E_0^2 \exp\left(-2\frac{r^2}{r_0^2}\right)$$

$\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377\ \Omega$ is the impedance of free space and n is the refractive index.

Therefore $P = \int_0^\infty 2\pi r \frac{1}{2}\frac{n}{\eta_0} E_0^2 \exp\left(-2\frac{r^2}{r_0^2}\right) dr = \pi \frac{n}{\eta_0} E_0^2 \left(-\frac{r_0^2}{4}\right) \int_0^\infty -\frac{4r}{r_0^2} \exp\left(-2\frac{r^2}{r_0^2}\right) dr = \pi \frac{n}{\eta_0} E_0^2 \frac{r_0^2}{4}$

$$V = \frac{2\pi \times 5}{1.48} \times \sqrt{1.45^2 - 1.446^2} = 2.28$$

Therefore $r_0 = \frac{5}{\sqrt{\ln 2.28}} = 5.5\ \mu\text{m}$

Therefore $\underline{P = \pi \times \frac{1.45}{377} \times 36 \times 10^{12} \times \frac{(5.5\times10^{-6})^2}{4} = 3.3}$ W

30 marks

$$P = \pi \frac{n}{\eta_0} E_0^2 \frac{r_0^2}{4}$$

And also

$$r_0 = \frac{a}{\sqrt{\ln V}}$$

So

$$P = \pi \frac{n}{\eta_0} E_0^2 \frac{a^2}{\ln V}$$

Scattering at $r = a$ is

$$S = E_0 \exp\left(-\frac{a^2}{r_0^2}\right)$$

To determine maximum consider S^2 such that

$$S^2 = E_0^2 \exp\left(-\frac{2a^2}{r_0^2}\right)$$

And

$$\frac{S^2}{P} = \frac{\exp(-2\ln V)}{\pi n a^2}\eta_0 \ln V = \frac{\eta_0}{\pi n a^2}\frac{\ln V}{V^2}$$

Hence

$$\frac{d}{dV}\left(\frac{S^2}{P}\right) = \frac{\eta_0}{\pi n a^2}\left[\frac{1}{V}\frac{1}{V^2} - \frac{2\ln V}{V^3}\right]$$

Therefore for maximum $1 - 2\ln V = 0$

Hence $V = \exp\left(\frac{1}{2}\right) = \sqrt{e} = 1.68$

30 marks