Worked solutions for 4B23 Optical Fibre Communication 2023/24 paper version 2

Question 1

a)

$$\frac{d}{dz}\mathbf{A} = -j\kappa J_3\mathbf{A}$$

Where $\mathbf{A} = [A_1, A_2, A_3]^T$

$$\frac{dA_1(z)}{dz} = -j\kappa[A_1(z) + A_2(z) + A_3(z)]$$

Let $A_S = A_1 + A_2 + A_3$

$$\frac{dA_s(z)}{dz} = -3j\kappa A_s(z)$$

which has solution $A_s(z) = e^{-3j\kappa z}A_s(0)$

[20]

b)

Using this

$$\frac{dA_1(z)}{dz} = -j\kappa e^{-3j\kappa z}A_s(0)$$

Hence integrating from $z = 0$ to $z = L$ gives

$$A_1(L) = -j\kappa A_s(0)\frac{e^{-3j\kappa L}}{-3j\kappa} + C_1 = \frac{e^{-3j\kappa L}}{3}A_s(0) + C_1$$

Where C_1 is a constant of integration. Applying the boundary condition that $A_1(0)$ at $z = 0$ gives

$$A_1(0) = A_s(0)/3 + C_1$$

Hence $C_1 = A_1(0) - A_s(0)/3$ i.e.

$$A_1(L) = \frac{e^{-3j\kappa L} - 1}{3}A_s(0) + A_1(0)$$

By symmetry the same solution applies for $A_2(L)$ and $A_3(L)$ such that for $i \in \{1,2,3\}$

$$A_i(L) = \frac{e^{-3j\kappa L} - 1}{3}A_s(0) + A_i(0)$$

[20]

c)

Consider an input [1,0,0]

Hence

$$A_1(L) = \frac{e^{-3j\kappa z} - 1}{3} + 1$$

$$A_2(L) = A_3(L) = \frac{e^{-3j\kappa z} - 1}{3}$$

$$|A_1(L)|^2 = \cos 3\kappa z - 1$$

If the input is $\mathbf{A}(0) = \left[A_{sig}, 0, A_{lo}\right]^T$

$$A_1(L) = \frac{e^{-3j\kappa z} + 2}{3}A_{sig} + \frac{e^{-3j\kappa z} - 1}{3}A_{lo}$$

and

$$A_3(L) = \frac{e^{-3j\kappa z} + 2}{3}A_{lo} + \frac{e^{-3j\kappa z} - 1}{3}A_{sig}$$

Hence

$$\frac{i_1}{R} = A_1(L)A_1^*(L) = \left|\frac{e^{-3j\kappa L} + 2}{3}A_{sig}\right|^2 + \frac{e^{-3j\kappa L} + 2}{3}A_{sig}\frac{e^{3j\kappa L} - 1}{3}A_{lo}^* + \left|\frac{e^{-3j\kappa L} - 1}{3}A_{lo}\right|^2 + c.c.$$

The term c_1 multiplying $A_{sig}A_{lo}^*$ is given by

$$c_1 = \frac{R}{9}\left(e^{-3j\kappa L} + 2\right)\left(e^{3j\kappa L} - 1\right) = \frac{R}{9}\left(2e^{3j\kappa L} - e^{-j3\kappa L} - 1\right) = \frac{R}{9}(\cos 3\kappa L - 1 + 3j \sin 3\kappa L)$$

And for i_3

$$\frac{i_3}{R} = A_3(L)A_3^*(L) = \left|\frac{e^{-3j\kappa L} + 2}{3} A_{lo}\right|^2 + \left|\frac{e^{-3j\kappa L} - 1}{3} A_{sig}\right|^2 + \frac{e^{-3j\kappa L} - 1}{3} A_{sig} \frac{e^{3j\kappa L} + 2}{3} A_{lo}^* + c.c.$$

Hence

$$c_3 = \frac{R}{9}\left(e^{-3j\kappa L} - 1\right)\left(e^{3j\kappa L} + 2\right) = \frac{R}{9}\left(2e^{-3j\kappa L} - e^{3j\kappa L} - 1\right) = \frac{R}{9}(\cos 3\kappa L - 1 - 3j \sin 3\kappa L)$$

[30]

d)

Considering the points on an Argand plane we deduce that 0 c_1 and c_3 form an isosceles triangle with 90 degrees at the origin which in turn requires

$$\cos(3\kappa L) - 1 = \pm 3 \sin(3\kappa L)$$

Therefore using standard trig identities

$$-2\sin^2\left(\frac{3\kappa L}{2}\right) = \pm 6 \sin\left(\frac{3\kappa L}{2}\right)\cos\left(\frac{3\kappa L}{2}\right)$$

Hence

$$\tan\left(\frac{3\kappa L}{2}\right) = \pm 3$$

Hence

$$\kappa L = \frac{2\pi m}{3} \pm \frac{2}{3}\tan^{-1}(3)$$

Hence

$$e^{-3j\kappa L} = e^{\mp 2j \tan^{-1}(3)} = -\frac{4}{5} \pm \frac{3j}{5}$$

Hence for an input [1,0,0]

The outputs are

$$\left[\frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3} + 1, \frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3}, \frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3}\right]$$

Hence the powers are

$$\left[\left|\frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3} + 1\right|^2, \left|\frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3}\right|^2, \left|\frac{-\frac{4}{5} \pm \frac{3j}{5} - 1}{3}\right|^2\right]$$

$$= \left[\left(\frac{2}{5}\right)^2 + \left(-\frac{1}{5}\right)^2, \left(-\frac{3}{5}\right)^2 + \left(-\frac{1}{5}\right)^2, \left(-\frac{3}{15}\right)^2 + \left(-\frac{1}{5}\right)^2\right] = \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]$$

So the power is split in the ratio 1:2:2

[30]

Question 2

a)

i) $\eta N_2 n$ represents stimulated emission, [2]

ii) $-\eta N_1 n$ represents absorption [2]

iii) ηN_2 represents spontaneous emission [2]

b)

$$\frac{dn(z)}{dz} = \eta(N_2 - N_1)n(z) + \eta N_2$$

$$\frac{d}{dz}\{n(z)e^{-\eta(N_2-N_1)z}\} = \eta N_2 e^{-\eta(N_2-N_1)z}$$

$$n(z)e^{-\eta(N_2-N_1)z} = -\frac{N_2}{N_2 - N_1}e^{-\eta(N_2-N_1)z} + C$$

For $z = 0$ we have $n(0) = n_{in}$ and hence

$$n_{in} = -\frac{N_2}{N_2 - N_1} + C$$

Therefore

$$n(z)e^{-\eta(N_2-N_1)z} = -\frac{N_2}{N_2 - N_1}e^{-\eta(N_2-N_1)z} + \left(n_{in} + \frac{N_2}{N_2 - N_1}\right)$$

Therefore, for a length of erbium fibre of length L defining $G = e^{\eta(N_2-N_1)L}$ to be the gain and $n_{sp} = N_2/(N_2 - N_1)$ the number of photons exiting will be

$$\frac{n(L)}{G} = -\frac{n_{sp}}{G} + (n_{in} + n_{sp})$$

therefore

$$n(L) = n_{in}G + n_{sp}(G - 1)$$

In a time T the number of photons will be

$$\frac{P_{out}T}{h\nu} = \frac{GP_{in}T}{h\nu} + n_{sp}(G - 1)$$

Noting that the equivalent bandwidth is $B = 1/(2T)$, such that $2B = 1/T$ gives

$$P_{out} = GP_{in} + 2Bh\nu n_{sp}(G - 1)$$

[24]

c)

i) Shannon capacity gives

$$C = 2B\log_2(1 + SNR)$$

$C = 1600$ Tbit/s, B =200 GHz

Hence

$$4 = \log_2(1 + SNR)$$

Therefore $SNR = 15$ which is 11.8 dB, hence maximum permitted NSR is -11.8 dB (1/15) [10]

ii) Initial SNR=23 dB which is 200, therefore NSR from the link is

1/15-1/200=37/600

From each EDFA the noise is

$N_{ASE} = 2h\nu B(G - 1) = 2 \times 1.3 \times 10^{-19} \times 200 \times 10^9 \times (G - 1) = 52(G - 1)$ nW

There are two EDFAs so total noise is $104(G-1)$ nW

Therefore require

$$\frac{10^6}{104(G-1)} = \frac{600}{37}$$

Hence

$$G = 1 + \frac{37 \times 10^6}{600 \times 104}$$

Therefore

$$G = 594$$

G=27.7 dB
Therefore with 0.15 dB/km
L=185 km so total system length is 370 km

[30]

iii) Transmitted pump is 2.5 W = 34 dBm
Received pump power is 34-Lx0.18 dBm
Total output power is 40% of the pump power, 40% is equivalent to -4 dB
Hence total output power is
30-Lx0.18 dBm
If we wanted 0 dBm per wavelength then total output power is 13.42 dBm hence

$$L = \frac{30 - 13.4}{0.18} = 92 \text{ km}$$

Which gives total system length of 184 km

However we can do better if we reduce the launch power

Total ASE power is

$$N_{ASE} = 2h\nu B(G-1) = 2 \times 1.3 \times 10^{-19} \times 4.4 \times 10^{12} \times (G-1) = 1.144 \times 10^{-6}(G-1) \quad \text{W}$$

Assuming $G - 1 \approx G$

And if gain matches fibre loss then $10\log_{10} G = 0.15\, L$ hence

$$10\log_{10}\left(\frac{N_{ASE}}{1mW}\right) = -29.4 + 0.15\, L$$

Therefore

$$SNR_{dB} = 30 - 0.18L - 3 - (-29.4 + 0.15L) = 56.4 - 0.33L$$

With 3 dB reduction due to two EDFAs.

But required link SNR is $10\log_{10}(600/37) = 12.1$ dB
Hence

$$12.1 = 56.4 - 0.33L$$

Which gives

$$L = 134.3\ km$$

Optimal EDFA output power is 5.8 dBm

So total reach is 268.6 km.
Doing a cross check
The total output power is $30 - 0.18L = 5.8$ dBm
Hence per channel power is 5.8-13.4=-7.6 dBm
Hence power is reduced by 7.6 dB c.f. previous part corresponding to 50.7 km of loss so expect L to reduce by 50.7 km (giving L=134.3 km) which agrees with answer.

[30]

Question 3

a) Equalisation stages are
- Coherent detection equalisation
- Static channel equalisation
- Adaptive channel equalisation

[10]

b) Rectangular Nyquist spectrum means that a 200 GBd signal will occupy 200 GHz of optical spectrum. At 1550 nm, 100 GHz occupies 0.8 nm so 200GHz corresponds to 1.6 nm and the chromatic dispersion from 400 km is

$$17 \times 1.6 \times 400 = 10880 \text{ ps}$$

The sampling rate is $200 \times \frac{64}{63} = 203.2$ GSa/s therefore the number of tap to span 10880 ps is

$$N_{CD} = 1 + 10880 \times 10^{-12} \times 200 \times 10^{9} \times \frac{64}{63} = 2212 \text{ taps}$$

[20]

c) Using the overlap and save algorithm with an N point FFT the number of complex multiplies per sample N_{cm} is

$$N_{cm} = \frac{N \log_2(N) + N}{N - N_{CD} + 1}$$

Given $N_{CD} = 2212$ we expect the minimum value of N to be 2^{12}=4096 which gives

$$N_{cm} = \frac{4096 \times 12 + 4096}{4096 - 2212 + 1} = 28.2$$

Given there are no technological limitations regarding the FFT size let us consider $N = 8192$ which gives

$$N_{cm} = \frac{8192 \times 13 + 8192}{8192 - 2212 + 1} = 19.2$$

Increasing to $N = 16384$ gives

$$N_{cm} = \frac{16384 \times (14 + 1)}{16384 - 2212 + 1} = 17.3$$

Increasing to $N = 2^{15}$ gives

$$N_{cm} = \frac{2^{15} \times (15 + 1)}{2^{15} - 2212 + 1} = 17.2$$

Increasing to $N = 2^{16}$ gives

$$N_{cm} = \frac{2^{16} \times (16 + 1)}{2^{16} - 2212 + 1} = 17.6$$

Hence the optimum value of N is 2^{15}. The power consumption per polarisation is

$P = 17.2 \times 0.1 \times 10^{-12} \times 203.2 \times 10^{9} = 0.35\ W$ and hence for two polarisations the total power consumption is 0.7 W.

[30]

d) PMD coefficient is 0.1 $\mathrm{ps}/\sqrt{\mathrm{km}}$ and therefore for 400 km the mean DGD is $\langle \Delta\tau \rangle = 0.1 \times \sqrt{400} = 2$ ps

From data sheet $P(\Delta\tau > x) \approx \frac{4}{\pi}\frac{x}{\langle \Delta\tau \rangle}\exp\left(-\frac{4x^2}{\pi \langle \Delta\tau \rangle^2}\right)$

With $\frac{x}{\langle \Delta\tau \rangle} = 4.8\ \ P(\Delta\tau > x) = 10^{-12}$

Hence span in samples is $4.8 \times 2 \times 10^{-12} \times 203.2 \times 10^{9} = 1.95$

Only need three taps so for the 2x2 MIMO equaliser total of 12 taps so 12 complex multiplications

Total power would be $12 \times 203.2 \times 10^{9} \times 0.1 \times 10^{-12} = 0.24$ W

[20]

e) The two polarisation chromatic dispersion compensating filter requires 0.7 W therefore the adaptive 2x2 MIMO filter which incorporates the chromatic dispersion will require 4 filters of length 2212 taps using a total of 1.4 W.

Therefore the transition will occur when the separate adaptive equaliser requires 0.7 W, hence per filter 8.6 complex multiplies per symbol.

A reasonable starting guess is 9 taps which with $N = 64$ requires $N_{cm} = 8$. If we assume $N = 64$ is optimal $N = 13$ gives $N_{cm} = 8.6$, hence for more than 13 taps combining the adaptive equaliser with the chromatic dispersion compensating filter will use less power. The drawback of this is however that the adaptation rate will decrease since 2^{15} samples are required for each block.

[20]