Worked solutions for 4B23 Optical Fibre Communication 2024/25 paper version 1

Question 1

a)

i) Since

$$\frac{dA_1}{dz} = -j\kappa A_2$$

Then

$$\frac{d^2A_1}{dz^2} = -j\kappa\frac{dA_2}{dz} = -j\kappa(-j\kappa A_1) = -\kappa^2 A_1$$

Therefore solution is

$$A_1(z) = \alpha\cos(\kappa z) + \beta\sin(\kappa z)$$

Substituting $z = 0$ gives $A_1(0) = \alpha$ so $A_1(z) = A_1(0)\cos(\kappa z) + \beta\sin(\kappa z)$

But from $\frac{dA_1}{dz} = -j\kappa A_2$ we note

$$A_2(z) = \frac{j}{\kappa}\frac{dA_1}{dz} = \frac{j}{\kappa}[A_1(0)(-\kappa)\sin(\kappa z) + \beta\kappa\cos(\kappa z)] = -jA_1(0)\sin(\kappa z) + j\beta\cos(\kappa z)$$

Therefore $A_2(0) = j\beta$ hence $\beta = -jA_2(0)$

Therefore

$$A_1(z) = A_1(0)\cos(\kappa z) - jA_2(0)\sin(\kappa z)$$

And

$$A_2(z) = -jA_1(0)\sin(\kappa z) + A_1(0)\cos(\kappa z)$$

Comparing with $\mathbf{A}(z) = S(z)\mathbf{A}(0)$ gives

$$S(z) = \begin{bmatrix}\cos(\kappa z) & -j\sin(\kappa z)\\ -j\sin(\kappa z) & \cos(\kappa z)\end{bmatrix}$$

[20]

ii) $S^H(z) = \begin{bmatrix}\cos(\kappa z) & j\sin(\kappa z)\\ j\sin(\kappa z) & \cos(\kappa z)\end{bmatrix}$

Therefore

$$S^H(z)S(z) = \begin{bmatrix}\cos(\kappa z) & j\sin(\kappa z)\\ j\sin(\kappa z) & \cos(\kappa z)\end{bmatrix}\begin{bmatrix}\cos(\kappa z) & -j\sin(\kappa z)\\ -j\sin(\kappa z) & \cos(\kappa z)\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2(\kappa z) - (j)^2\sin^2(\kappa z) & -j\sin(\kappa z)\cos(\kappa z) + j\sin(\kappa z)\cos(\kappa z)\\ -j\sin(\kappa z)\cos(\kappa z) + j\sin(\kappa z)\cos(\kappa z) & \cos^2(\kappa z) - (j)^2\sin^2(\kappa z)\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

As required

Alternative approach $\mathbf{A^H}(z) = \mathbf{A^H}(0)S^H(z)$

Hence $|A_1(z)|^2 + |A_2(z)|^2 = \mathbf{A^H}(z)\mathbf{A}(z) = \mathbf{A^H}(0)S^H(z)S(z)\mathbf{A}(0)$

Power conservation requires $\mathbf{A^H}(z)\mathbf{A}(z) = \mathbf{A^H}(0)\mathbf{A}(0)$ hence $S^H(z)S(z) = I$

[10]

iii) Comparing forms give $\sqrt{1-R} = \cos(\kappa z)$ and $a = -j\sin(\kappa z)$

But since $\sin^2 x + \cos^2 x = 1$ then $a = -j\left[\pm\sqrt{1-(1-R)}\right] = \pm j\sqrt{R}$

[15]

b) The 3x3 coupler will also conserve power so $S_3^H S_3 = I_3$ where I_3 is the 3x3 identity matrix

$$S_3 = \begin{bmatrix} \sqrt{1-R} & b & b \\ b & \sqrt{1-R} & b \\ b & b & \sqrt{1-R} \end{bmatrix}$$

and

$$S_3^H = \begin{bmatrix} \sqrt{1-R} & b^* & b^* \\ b^* & \sqrt{1-R} & b^* \\ b^* & b^* & \sqrt{1-R} \end{bmatrix}$$

Hence considering the diagonal and off-diagonal terms on the identity matrix gives

$$1 = 1 - R + 2|b|^2$$

and

$$0 = b\sqrt{1-R} + b^*\sqrt{1-R} + |b|^2$$

Therefore

$$R = 2|b|^2 = -2(b + b^*)\sqrt{1-R}$$

Let $b = x + jy$

$$R = -4x\sqrt{1-R}$$

Hence

$$x = -\frac{R}{4\sqrt{1-R}}$$

But $R = 2|b|^2 = 2x^2 + 2y^2$ therefore

$$R = \frac{2R^2}{16(1-R)} + 2y^2$$

Hence

$$2y^2 = \frac{16(1-R)R - 2R^2}{16(1-R)} = \frac{16R - 18R^2}{16(1-R)}$$

Therefore

$$y = \pm\frac{\sqrt{8R - 9R^2}}{4\sqrt{1-R}} = \mp\frac{x}{R}\sqrt{8R - 9R^2}$$

Therefore

$$b = -\frac{R}{4\sqrt{1-R}}\left(1 \mp j\sqrt{\frac{8}{R} - 9}\right)$$

We require $\frac{8}{R} - 9 \geq 0$ hence $R = 8/9$.

[25]

c) If we consider 2x2 couplers we need 5 stages since $2^5 = 32$, and there will be 1+2+4+8+16=31 couplers

Total loss of 5x3.5=17.5 dB

If we consider 3x3 couplers we need 3 stages since $3^3 = 27$ and there will be 1+3+9=13 couplers

Total loss of 3x5.8=17.4 dB

If we use 2x2 and 3x3 couplers we can use 4 stages since 24=2x2x2x3 so the total loss is 3.5x3+5.8=16.3 dB
To minimise the number of couplers we use the 3x3 coupler at the final stage so there will be 1+2+4+8=15 couplers – 8 of which are 3x3 and 7 are 2x2. Had we used the 3x3 coupler for the first stage there would be 1+3(1+2+4)=22 couplers!

[30]

Question 2

a) Rectangular Nyquist spectrum means that a 240 GBd signal will occupy 240 GHz of optical spectrum. At 1550 nm, 100 GHz occupies 0.8 nm so 240GHz corresponds to 1.92 nm and the chromatic dispersion from 120 km is

$$17 \times 1.92 \times 120 = 3916.8 \text{ ps}$$

The sampling rate is $240 \times \frac{8}{7} = 274.3$ GSa/s therefore the number of tap to span 2611.2 ps is

$$N_{CD} = 1 + 3916.8 \times 10^{-12} \times 240 \times 10^9 \times \frac{8}{7} = 1076 \text{ taps}$$

[25]

b) Using the overlap and save algorithm with an N point FFT the number of complex multiplies per sample N_{cm} is

$$N_{cm} = \frac{N \log_2(N) + N}{N - N_{CD} + 1}$$

Given $N_{CD} = 1076$ we expect the minimum value of N to be $N = 2^{12} = 4096$ which gives

$$N_{cm} = \frac{4096 \times 12 + 4096}{4096 - 1076 + 1} = 17.6$$

Given there are no technological limitations regarding the FFT size let us consider $N = 8192$ which gives

$$N_{cm} = \frac{8192 \times 13 + 8192}{8192 - 1076 + 1} = 16.11$$

Increasing to $N = 16384$ gives

$$N_{cm} = \frac{16384 \times (14 + 1)}{16384 - 1076 + 1} = 16.05$$

Increasing to $N = 32768$ gives

$$N_{cm} = \frac{32768 \times (15 + 1)}{32768 - 1076 + 1} = 16.5$$

Hence the optimum value of N is 16384. The power consumption per polarisation is

$P = 16.05 \times 0.1 \times 10^{-12} \times 240 \times 10^9 \times \frac{8}{7} = 0.44\ W$ and hence for two polarisations the total power consumption is 0.88 W.

[25]

c)

Transceiver NSR is -15.2 dB=0.0302

Maximum NSR is -13.6 dB=0.0437

Hence link NSR 0.0437-0.0302=0.0135

From the EDFA the noise power is

$$\sigma_{ASE}^2 = 2n_{sp}hv(G-1)B = 2 \times 2 \times 1.3 \times 10^{-19} \times 250 \times 240 \times 10^9 = 3.12 \times 10^{-5} = 31.2\ \mu\text{W}$$

Hence transmitted signal power is

$$\frac{31.2 \times 10^{-6}}{0.0135} = 2.3\ \mathrm{mW} = 3.6\ \mathrm{dBm}$$

[25]

d)

With nonlinearity $NSR = \frac{C_{NLI} G_{Tx}^3 + N_{ASE}}{G_{Tx}} = \frac{C_{NLI}}{R^2} \times P^2 + \frac{\sigma_{ASE}^2}{P}$

$$C_{NLI} = \frac{8\gamma^2 L_{eff}^2 \alpha}{27\pi|\beta_2|} \ln\left(\frac{|\beta_2|}{\alpha}\pi^2 B^2\right)$$

Where $B = R = 0.24$ THz hence noting for 120km,

$$L_{eff} \approx \frac{1}{\alpha} = \frac{1}{0.23 \times 0.2} = 21.7\ \mathrm{km}$$

and

$$\beta_2 = -\frac{17}{0.78} = -21.8\ \mathrm{ps}^2/\mathrm{km}$$

hence

$$C_{NLI} = \frac{8 \times 1.3^2 \times 21.7}{27 \times \pi \times 21.8} \ln\left(\frac{21.8}{1/21.7} \times \pi^2 \times 0.24^2\right) = 0.8876\ (pJ)^{-2}$$

Hence

$$NSR_{NLI} = \frac{C_{NLI}}{R^2} \times P^2 = \frac{0.8876}{0.24^2} \times (2.3 \times 10^{-3})^2 = 8.2 \times 10^{-5}$$

Since this is much less than the $NSR_{ASE} = 0.0135$ the assumption is valid.

N.B. the optimum launch power is

$$P_{opt} = R \times G_{opt} = R \times \sqrt[3]{\frac{N_{ASE}}{2C_{NLI}}} = R \times \sqrt[3]{\frac{\sigma_{ASE}^2/R}{2C_{NLI}}} = 0.24 \times \sqrt[3]{\frac{3.12 \times 10^{-6}/0.24}{2 \times 0.8876}} = 0.01\ W = 10\ \mathrm{dBm}$$

Hence with a launch power 6 dB below the optimum linear performance is expected.

[25]

Question 3

a) The cut-off wavelength is the wavelength above which only the LP_01 mode propagates. This occurs when $V = 2.405$ and hence

$$\lambda_c = \frac{2\pi a}{2.405}\sqrt{n_{co}^2 - n_{cl}^2} = \frac{2\pi \times 4 \times 10^{-6}}{2.405}\sqrt{1.45^2 - 1.443^2} = 1487 \text{ nm}$$

[15]

b) We can use Shannon capacity to estimate this. The start of the C band is 1530 nm which corresponds to a frequency of $3 \times 10^8/1.53 \times 10^{-6} = 196.1$ THz. The end of the U band is 1675 nm which corresponds to a frequency of $3 \times 10^8/1.675 \times 10^{-6} = 179.1$ THz. Hence the total bandwidth is 17 THz

Therefore

$$C = 2 \times 17 \times 10^{12} \times \log_2(1 + 10^{2.7}) = 305 \text{ Tbit/s}$$

[20]

c) The attenuation is given by

$$\alpha_{dB} = \frac{1}{\lambda^4} + \exp\left(50\left[\frac{1}{1.8} - \frac{1}{\lambda}\right]\right)$$

Differentiating w.r.t. λ gives

$$\frac{d\alpha_{dB}}{d\lambda} = -\frac{4 \times 1}{\lambda^5} + \frac{50}{\lambda^2}\exp\left(50\left[\frac{1}{1.8} - \frac{1}{\lambda}\right]\right)$$

Setting the derivative equal to zero gives

$$\frac{2}{25\lambda^3} = \exp\left(50\left[\frac{1}{1.8} - \frac{1}{\lambda}\right]\right)$$

i.e.

$$\ln(2/25) - 3\ln\lambda = 50\left[\frac{1}{1.8} - \frac{1}{\lambda}\right]$$

Using the solver function on the calculator (fx-991CW) with an initial estimate of 1.5 gives λ =1.578623 µm i.e. $\lambda = 1578.6$ nm

Evaluating the function a_{dB} at this wavelength gives the minimum attenuation as $\alpha_{dB} = 0.181$ dB/km.

[25]

d) To estimate the capacity we assume Shannon capacity

$$C = 2B\log_2(1 + SNR)$$

And assume shot noise is the fundamental limitation so $SNR = \frac{P}{2h\nu B_{eff}}$

at 1550 nm $h\nu = 0.8$ eV, hence from 1530 nm to 1675 nm varies from 0.81 to 0.74 eV.
Between 1530 nm and 1675 nm attenuation varies between 0.182 dB/km and 0.253 dB/km with a minimum attenuation is 0.181 dB/km (from previous part)

Initial power is 1W = 30 dBm. Hence P varies between 30-0.253x25=23.7 dBm=234 mW and 25.5 dBm=355 mW

Hence worst-case capacity with $h\nu = 0.81$ eV and P=234 mW

Giving

$$C = 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{0.234}{2 \times 0.81 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) = 534 \times 10^{12} = 534 \text{ Tbit/s}$$

Likewise best-case capacity with $h\nu = 0.74$ eV and P=355 mW

$$C = 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{0.355}{2 \times 0.75 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) = 557 \times 10^{12} = 558 \text{ Tbit/s}$$

Hence to one significant figure we estimate the capacity to be 500 Tbit/s (taking the lower of the two estimates)

[20]

e) This final part has a subtly – since the O-band is below the cut-off wavelength so will support two modes! Roughly the bandwidth of the O-band and C-U band are similar so we assign 1/3 W to C-U band and 1/3 W to each of the two modes in the O-band.

From the previous section we take

$$C_{CU} = 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{0.234/3}{2 \times 0.81 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) = 480 \text{ Tbit/s}$$

In the O-band 1260 nm =238 THz and 1360 nm =221 THz hence B=17 THz

1550 nm $h\nu = 0.8$ eV, hence from 1260 nm to 1360 nm varies from 0.98 to 0.91 eV.
Between 1260 nm and 1360 nm attenuation varies between 0.397 dB/km and 0.292 dB/km

Initial power per mode is 1/3 W = 25.2 dBm. Hence P varies between
25.2-0.397x25=15.3 dBm=34 mW and 25.2-0.292x25=17.9 dBm=62 mW

Taking the worst-case we get for each mode

$$C_O = 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{0.034}{2 \times 0.98 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) = 430 \text{ Tbit/s}$$

Hence total capacity of the two modes and the C-U band is 1340 Tbit/s i.e. more than 1 Pbit/s!

Aside: It turns out the guess of splitting the power is optimal – if we have a split ratio r to the CU band then each mode of the O band receives $\frac{1-r}{2}$

$$\begin{aligned} C_T &= C_{CU} + 2C_O \\ &= 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{0.234r}{2 \times 0.81 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) \\ &\quad + 2 \times 2 \times 17 \times 10^{12} \times \log_2\left(1 + \frac{\frac{0.102(1-r)}{2}}{2 \times 0.98 \times 1.6 \times 10^{-19} \times 17 \times 10^{12}}\right) \\ &= 34\log_2(53105r) + 68\log_2(9566[1-r]) = 534 + 899 + 34\log_2(r[1-r]^2) \\ &= 1433 + 34\log_2(r[1-r]^2) \end{aligned}$$

To get the optimal value of r we consider $\frac{d}{dr}\ln(r[1-r]^2) = \frac{d}{dr}\ln(r - 2r^2 + r^3) = \frac{1-4r+3r^2}{r-2r^2+r^3} = 0$
Hence

$$r_{opt} = \frac{4 \pm \sqrt{4^2 - 4 \times 3}}{2 \times 3} = \frac{4 \pm 2}{6} = \frac{1}{3}$$

[20]