

4B5 2022 solutions

1.

- (a) The answer should include a description of (i) the photoelectric effect and (ii) Compton scattering. The photoelectric effect was observed by Lenard in 1902 who showed that when light was incident on a metal surface, the energy of the emitted electrons increased with the frequency of the light. In 1905, Einstein then used Planck's theory of black-body radiation to describe light as consisting of quanta whose energy scaled with frequency. In 1916, Millikan performed detailed experiments which tied up Lenard's previous results with Einstein's ideas. Answer might include a sketch of KE vs frequency, with a mention of work function.

However, although we could describe light as comprising photons, it was not until Compton explored the scattering of X rays from materials that this was further developed. His observation was that the scattered light also contained a component with longer wavelength, where the change in wavelength depended on the scattering angle. This could only be explained by assuming that light comprises packets with a definite energy ($h\nu$) and momentum (h/λ), which is consistent with a particle.

Answer might include a sketch of scattered spectrum for a few different angles.

- (b) From the formula sheet, the wavelength of the secondary peak in Compton scattering is that $\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$. The shift in wavelength is therefore $2.426 \times 10^{-12} \times 0.5 \text{ m} = 1.213 \text{ pm}$. Therefore, the wavelength of the second peak will be $0.8787 \times 10^{-11} \text{ m}$. This will be observed as well as a larger peak at the original wavelength of $1 \times 10^{-11} \text{ m}$.
- (c) The discussion should focus on the fact that the KG equation is second-order in both space and time, like the wave-equation, whereas the Schrödinger equation is second order in space and first order in time. Therefore, the interpretation of the solutions is different, as the wave function squared from the KG equation does not represent a probability density. The KG equation also gives rise to negative energy solutions which cannot happen with the Schrödinger equation (except in tunnelling).

- (d) The wavefunction itself has no physical significance or meaning, but its square represents the result of a measurement of position, i.e. $|\psi(x)|^2$ yields the probability of the object being located at position x .

2.

- (a) Schrödinger's equation can be written in the regions to the left and right of the step as

$$(-\hbar^2/2m\partial^2/\partial x^2) \Psi_I(\mathbf{x}) = E\Psi_I(\mathbf{x}) \quad \text{Region I}$$

$$(-\hbar^2/2m\partial^2/\partial x^2 + V) \Psi_{II}(\mathbf{x}) = E\Psi_{II}(\mathbf{x}) \quad \text{Region II}$$

The solutions to these equations are:

$$\Psi_I(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \text{ where } k_1 = \frac{\sqrt{2mE}}{\hbar}$$

$$\text{and } \Psi_{II}(x) = A_2 e^{-k_2 x} \text{ where } k_2 = \frac{\sqrt{2m(V-E)}}{\hbar}$$

Matching the wave-functions and their first derivatives at the boundary ($x = 0$) yields the following relationships:

$$A_1 + B_1 = A_2$$

$$ik_1 A_1 - ik_1 B_1 = ik_2 A_2$$

$$\text{i.e. } B_1/A_1 = (k_1 - k_2)/(k_1 + k_2)$$

The reflection probability is $|B_1/A_1|^2$, which in this case equals **0.029**. This represents the probability that a given particle will be reflected from the potential step, so must lie between 0 & 1. R does *not* represent the fraction of a particle which will be reflected. If we pass a large number of electrons over this potential step, then on average, a fraction R of them will be reflected, but any individual electron will either be totally reflected or totally transmitted.

(b)

When there is a different effective mass in both regions, then we must use the BenDaniel-Duke boundary conditions, which are (i) wave-functions and (ii) derivatives of the wave-functions divided by their effective mass must match at the interfaces, i.e. :

$$A_1 + B_1 = A_2$$

$$i \frac{k_1}{m_1} A_1 - i \frac{k_1}{m_1} B_1 = i \frac{k_2}{m_2} A_2$$

i.e.

$$\frac{B_1}{A_1} = \frac{\frac{k_1}{m_1} - \frac{k_2}{m_2}}{\frac{k_1}{m_1} + \frac{k_2}{m_2}}$$

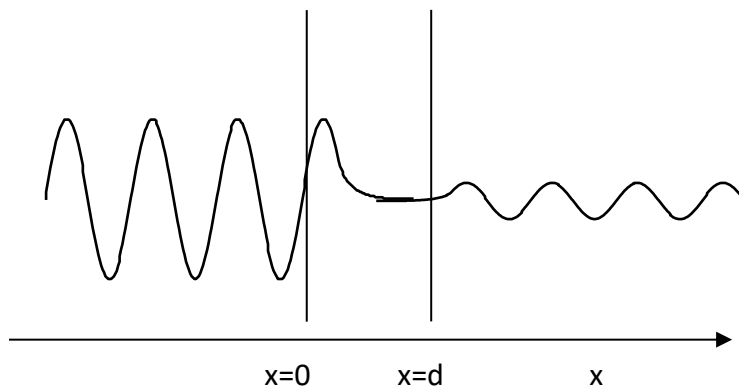
Given that $m_1 = 2.5m_2$, this reduces down to :

$$\frac{B_1}{A_1} = \frac{k_1 - 2.5k_2}{k_1 + 2.5k_2}$$

The reflection probability is $|B_1/A_1|^2$, which in this case equals **0.003**, which is almost a factor of 10 lower than what we found when it was the same material on both sides of the junction.

Examiner's note – most students did not use the BenDaniel-Duke boundary conditions, and therefore overestimated the reflection probability. If one makes this error, the calculated R is 0.146.

(c). Probability density:



If we say that the amplitude of the incident wave is 1, then the wave-function at the right hand edge of the barrier region is approximately: $\Psi_{II}(d) = e^{-k_2 d}$, so the transmission probability, T is roughly $e^{-2k_2 d}$. We are using the assumption that the coefficient $A_2 \sim 0$, as it is always $\ll B_2$.

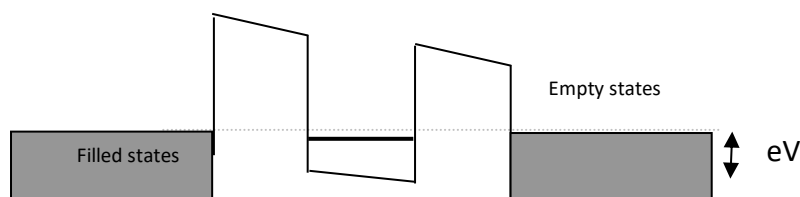
Given the values of E , and V , we obtain a value for $k_2 = 1.15 \times 10^9 \text{ m}^{-1}$. Therefore, the value of d at which the transmission probability is 20% is given by solving $0.2 = e^{-2k_2d}$ from which we find (by taking the ln of both sides) that $1.61 = 2k_2d \Rightarrow d = 0.7 \text{ nm}$.

Assumptions: (i) only an exponentially decaying term in barrier, (ii) effective mass of electron in each region is the same

To improve precision, would include full form of ψ_{II} taking into account the exponentially-increasing term, and would determine transmission probability for entire structure.

3.

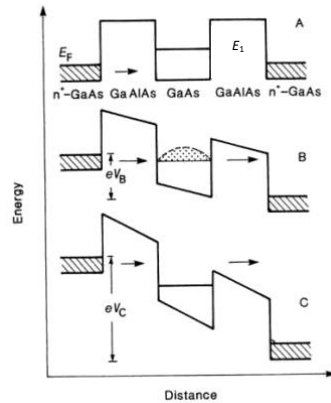
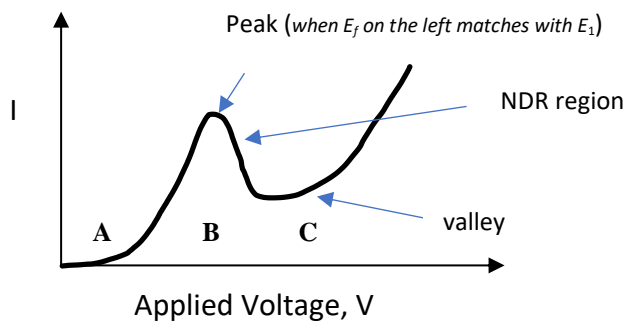
(a) This is a resonant tunnelling device. When a voltage is applied, the entire potential profile becomes sloped, and the conduction band edge on the left approaches bound state E_1 .



(b) The current, I depends on the transmission probability, T as follows:

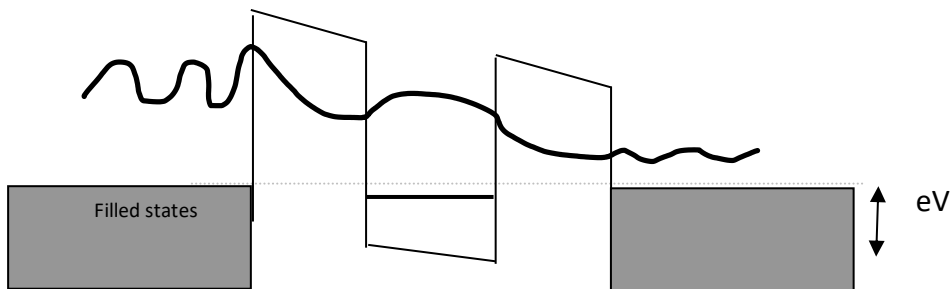
$$J_{\text{left}} = \frac{2e}{h} \int_0^{\infty} D(E) f(E) T(E) dE$$

As V is increased, the bound state energy E_1 starts below E_f , then when V is large enough they coincide, causing a peak in T . As V is further increased, there is no longer a match between E_1 and E_f , and T starts dropping. As we continue to increase V , eventually Thermal current will start to dominate and the conventional exponential increase in current with applied voltage for a diode will be observed.



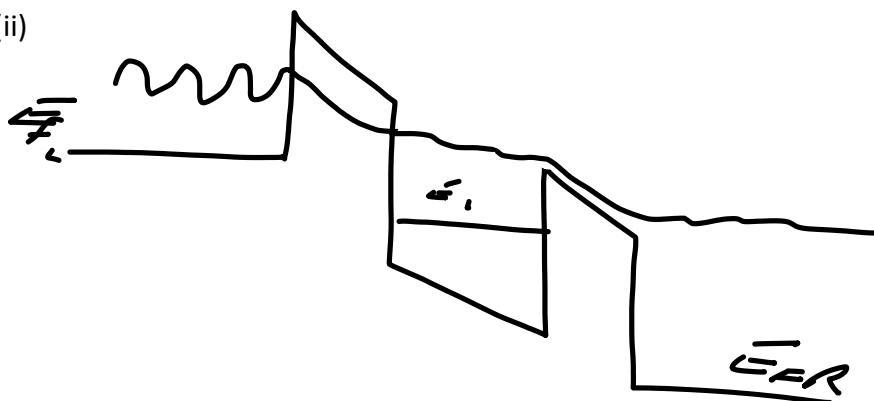
(c)

(i)



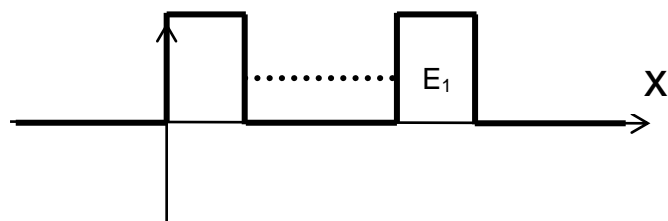
There is a resonant state inside the well, i.e., the length of the well is a half-integer number of electron wavelengths long. The wavefunction is therefore that for the ground state in a quantum well.

(ii)

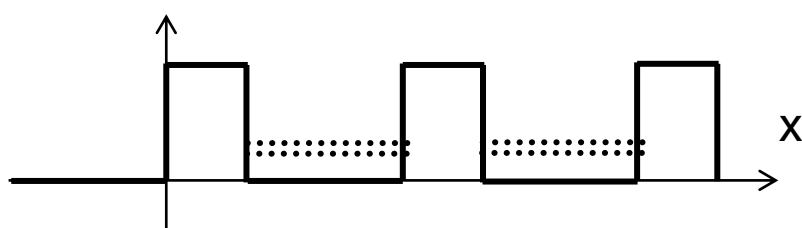


More challenging to draw, but the key point is the length of the well is no longer a half-integer number of electron wavelengths long, so there are no bound states, and the electron travels through it incoherently.

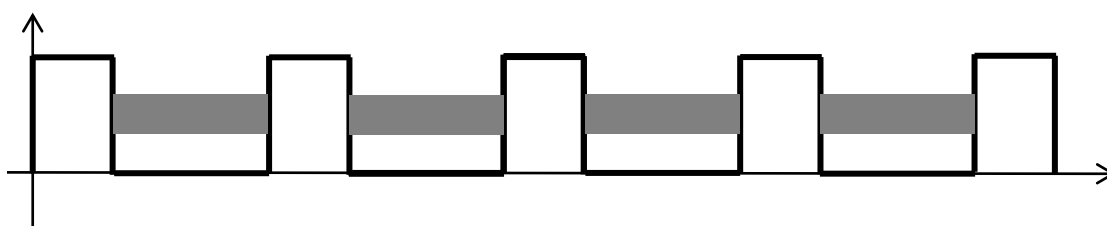
(d)



Now add in another well:



Now for a very large number of wells:



Energy levels become bands.

If the wells are close enough, the tail of the wave-functions within each well can extend appreciably into neighbouring wells. This gives rise to a coupling between the wells, and generates new wave-functions which are combinations of the original ones. For two wells, those combinations are the sum and difference between the original functions. Hence, there will be two states instead of one. For N wells, each state will become N closely spaced states. The closer the wells are to each other, the larger will be the splitting in energy. This is similar to beats in the addition of waves: the closer the frequency, the more pronounced are the beats.

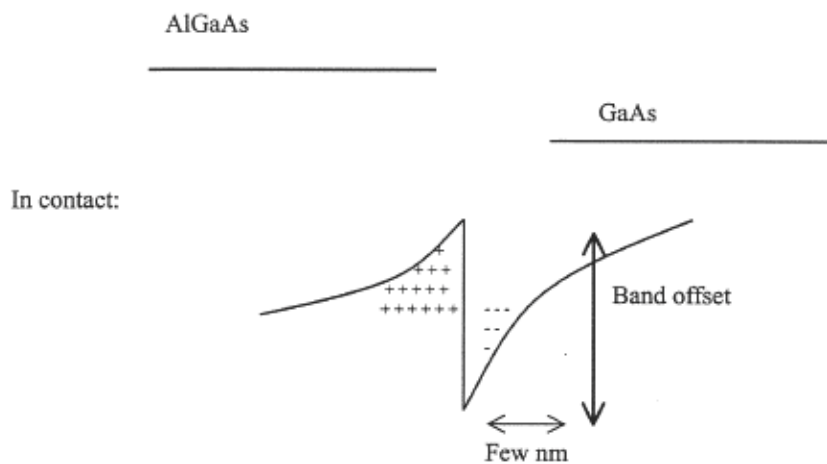
As the wells are brought closer, the coupling increases and so therefore does the splitting of the energy levels.

4.

(a) Band Engineering is when we combine semiconductors with different band gaps (in multilayer or superlattice structures) in order to create a potential profile with a specific electrical or optical property. Examples are quantum wells and resonant tunnelling devices.

(b) A heterojunction is a physical junction between two different materials. Consider what happens when we take a piece of GaAs and dope it on one side to make AlGaAs (which will therefore be *n*-type). GaAs has a lower band gap, and this will give rise to a band offset, which will cause some electrons to flow from the AlGaAs into the GaAs. As a result of this, the AlGaAs and the GaAs will be positively and negatively charged, respectively. This dipole layer of charge causes the bands to bend in the vicinity of the interface and creates two sheets of charge that are parallel to the interface.

i.e. Conduction bands before contact:

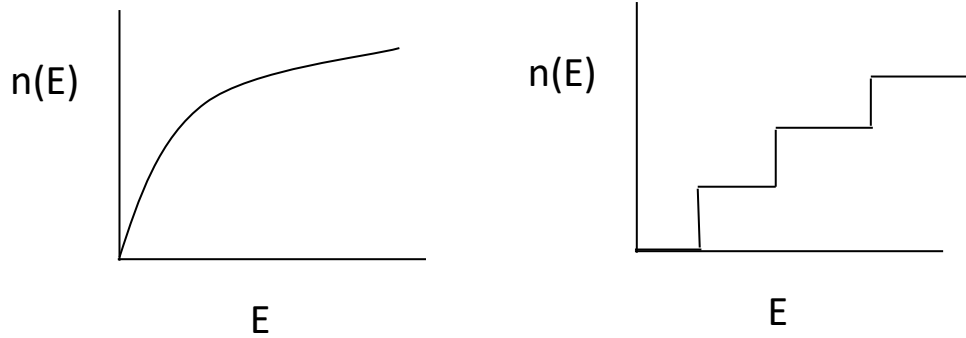


The 2-DEG is different in terms of its density of states:

3D
 $E^{1/2}$

2D

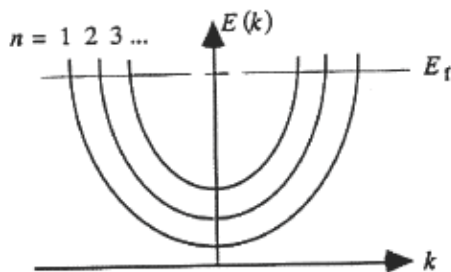
const.



The individual steps/plateaus correspond to sub-bands.

Electrons in a 2-DEG are confined in one direction (z) and free to move in the other two (x, y) parallel to the interface. The energy of these electrons are given by:

Where the E_n are the discrete energy eigenstates of the potential well. For most purposes, the triangular well can even be approximated as a square well. The x, y components of energy are continuous but the E_n are discrete. For each E_n , there will be a free-electron parabola in the $x-y$ plane, and this manifests as sub bands. The form is as shown below:



(c)

The emission wavelength will be the energy of the transition from the ground state electron state to the corresponding hole state plus the band gap.

Assume that well is infinitely deep (assumption 1) and square (assumption 2). Therefore, we can write energy eigenvalues as :

$$E_n = \frac{h^2 n^2}{8ml^2} \quad \text{where } l \text{ is the width of the well, and } m \text{ is the electron mass within the well.}$$

For the Valence band, then, $E_{h1} = 0.0124$ eV, and for the Conduction band $E_{e1} = 0.0623$ eV

- ⇒ emission wavelength of laser = $1.1 \text{ eV} + 0.0124 \text{ eV} + 0.0623 \text{ eV} = 1.1747 \text{ eV}$.
- ⇒ $1.17475 \text{ eV} = 1.88 \times 10^{-19} \text{ J} = hc/\lambda \Rightarrow \lambda = 1.06 \mu\text{m}$. The accuracy of this answer depends on the depth of the potential well, i.e. can we assume that it is infinitely deep? In reality, the well will be more likely to be less than 1 eV deep, so an energy level of 0.25 eV is not far below that, meaning the accuracy will be questionable. In order to improve on the accuracy, we should calculate the bound state energies of the finite well, taking into account that (i) it is finite and (ii) triangular.

(d) If we were to use a Quantum dot, the spread in laser wavelengths would be much smaller – it would be closer to monochromatic. Also, the threshold current would be lower.

Examiner's note – quite a few students mixed up a quantum dot with a single-electron transistor

5.

(a). The answer should include a sketch of how conductance should drop with size, and will scale with area as long as the cross-sectional dimensions are larger than the mean free path. For smaller dimensions, the conductance will start to drop more rapidly due to surface and grain-boundary scattering (answer should mention link between thickness of a film and mean grain size). Then, as dimensions approach $\sim 5 \text{ nm}$, quantisation of conductance should start to become apparent. The point about room temperature is related to the Kubo gap, i.e. when the spacing between the ground state and first excited state is greater than $k_B T$. By thinking of a wire as having hard walls - i.e. electrons are essentially in a potential well of height ϕ and width D , where ϕ and D are the work function of the metal and the diameter of the wire, respectively. We can make the approximation that the first level is well below ϕ and can therefore use the energy spectrum for the infinite potential well. This gives a Kubo gap of :

$$E_2 - E_1 = 3 \frac{h^2}{8mD^2}$$

For this to be equal to $k_B T$, we find that

$$D = \frac{h}{2} \sqrt{\frac{3}{2mk_B T}} = 0.66 \text{ nm}$$

(b). Answer should focus on 2DEG and how one would add contacts. The key points are the high mobility is achieved through injection of charge without doping, and that there needs to be a spacer layer to prevent scattering from the dopants in the adjacent layers. Mention Coulomb drag. Concept of hot electrons should be introduced and then there should be a sketch of a device where a potential step down is added, but pointing out that the hot region is only approx. 1 mean free path long.

(c) The answer should point out that the E-k relationship is linear rather than parabolic, which is consistent with relativistic but massless particles, suggesting they will travel ballistically. The lack of a band gap limits the on/off ratio to be in the range 1-10, whereas a meaningful device should have a ratio in excess of 10^6 . This can be addressed by somehow inducing a band gap, either by using a nanoribbon, or by creating an asymmetry in a bilayer device.