

1. (a) Quantum mechanically, the state of a particle is described by its wave-function,  $\psi$ , with the following characteristics:

Postulate 1:

The state of a quantum particle at time  $t$  is described by a continuous, non-singular, complex wave-function  $\psi(r,t)$ , which can be normalised so that the square of its modulus is equal to the probability density for the results of a position measurement.

Postulate 2:

A dynamical variable is represented by a linear Hermitian operator  $\hat{A}$  whose real eigenvalue spectrum is the set of all possible results of a measurement of that variable. The eigenfunction corresponding to a particular eigenvalue  $a$  describes an eigenstate of the quantum system, on which a measurement of the variable represented by  $\hat{A}$  yields the value  $a$  with probability 1.

A free particle propagating along  $r$  can be described as  $\psi(r,t) = e^{i(kr - \omega t)}$ , where  $\omega$  = angular frequency (=2 $\pi$ f) and  $k$  = wavenumber (=2 $\pi$ / $\lambda$ ).

(b) Energy operator,  $\hat{E} = i\hbar \frac{\partial}{\partial t}$   
Momentum operator,  $\hat{p} = -i\hbar \frac{\partial}{\partial r}$   
Kinetic Energy operator,  $\frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2}$

To every observable quantity is associated a corresponding operator.

Given a state described by a wavefunction  $\psi(x)$ , we can calculate the expectation value of any observable quantity in that state by using the corresponding operator:

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x) \hat{A} \psi(x) dx$$

$$\begin{aligned} [\hat{X}, \hat{P}] \psi(x) &= x(-i\hbar) \frac{d}{dx} \psi(x) - (-i\hbar) \frac{d}{dx} [x\psi(x)] \\ &= i\hbar \psi(x) \end{aligned}$$

- (c) Time-dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \psi(r,t) + V(r)\psi(r,t) = i\hbar \frac{\partial}{\partial t} \psi(r,t)$$

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

The wave equation describes the propagation of classical waves (sound waves, light waves, ...) while Schrödinger equation describes “quantum waves” or the time-evolution of objects following quantum mechanical rules.

The wave equation has a second time derivative as opposed to Schrödinger equation that contains a first time derivative only. Both have second time derivatives as a function of space.

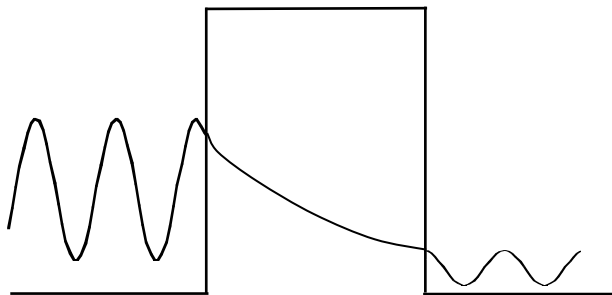
(d) The wave-function describing a particle must be a continuous function (postulate 1): when we have several boundaries separating different regions of space, the wave-function will have a different form in each region, but these functions must match at the boundaries (this is linked to probability conditions).

The first spatial derivatives must also be continuous at the boundaries (this is linked to conservation of energy).

Using these conditions, we can solve for the unknowns in the wave-function which we get from solving Schrödinger equation. Then we should normalise the wave-function.

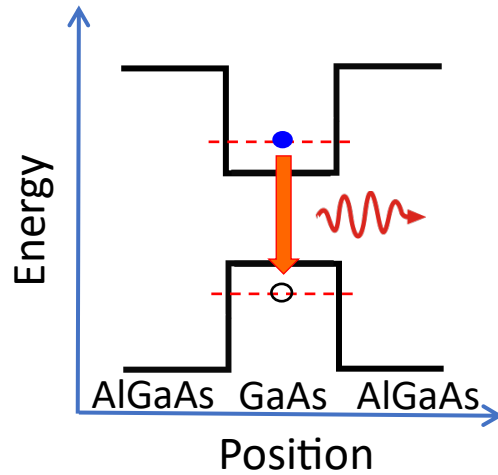
When going across an interface with different effective masses, then the first spatial derivatives divided by the mass must also be continuous at the boundaries – “BenDaniel-Duke boundary conditions” (this is linked to current continuity).

(e) In the quantum mechanical description, the electron has a non-zero probability of tunnelling through the barrier. The electronic wave will experience an exponential decay and, if the barrier is thin enough such that the wave intensity is non-zero at the second edge of the potential, the wave will propagate on the other side of the barrier.



Classically, we would expect zero probability of passing through (because the electron does not have enough energy to overcome the barrier).

2. (a)



(b) GaAs bandgap = 1.424 eV

$\lambda = 850 \text{ nm}$  corresponds to 1.46 eV

$[(1.46 - 1.424) / 2] \text{ eV} = 0.018 \text{ eV}$   $\rightarrow$  electron (and hole, given the assumptions)

confining potential

The infinite well approximation is valid when the bound states are deeply confined in the well, potential well height  $\gg$  than confinement energy.

Here  $(1.92 - 1.424 = 0.5 \text{ eV})$ , hence  $0.5/2 = 0.25 \text{ eV} \gg 0.018 \text{ eV}$

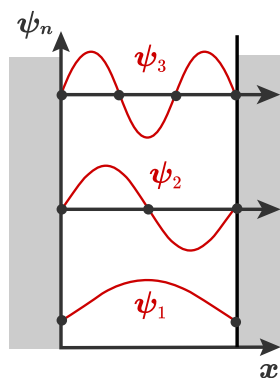
(c)  $E = \frac{h^2 n^2}{8mL^2} = 0.018 \text{ eV}$  for  $n=1$ .

Plugging in the numbers,  $L = 4.6 \text{ nm}$

(d) Confining potential depth =  $[(1.92 - 1.424) / 2] \text{ eV} = 0.248 \text{ eV}$

For  $n = 4$   $E = 0.288 \text{ eV}$ , for  $n = 3$ ,  $E = 0.162 \text{ eV}$ , for  $n = 2$ ,  $E = 0.072 \text{ eV}$   $\rightarrow$  3 bound states

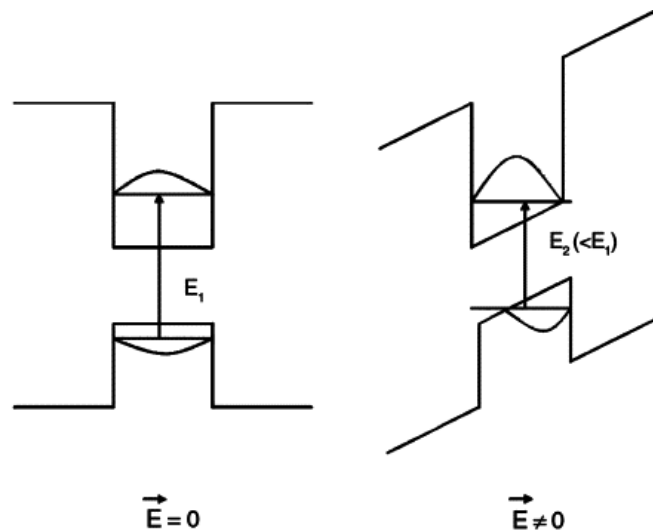
(e)



The wavefunction distributions are related to the probability of finding an electron within the quantum well, probability given by their modulus squared.

They have alternating symmetry, like confined waves.

(f)



If  $E$  applied in the opposite orientation, the bands will bend in the opposite direction.

(g) The confined energy states are brought closer together, therefore the energy separation is smaller. This results in a red-shift of the laser emission wavelength.

3. (a) The phonons are collective oscillations of the atoms in a lattice. The confinement potential can be described as parabolic and the system can be described as an harmonic oscillator.

Their energy, under the stated approximations, is given by:

$$E_n = (n + \frac{1}{2}) \hbar\omega$$

(b)  $\nu = 10$  THz

$$k_B T = \text{thermal energy} = (1.38 \cdot 10^{-23} \text{ J K}^{-1}) * T$$

$$E_2 = 2.5 h \cdot 10 \text{ THz} = 2.5 \cdot 6.63 \cdot 10^{-21} \text{ J} = 16.4 \cdot 10^{-21} \text{ J} = 1.64 \cdot 10^{-20} \text{ J}$$

$$T = (1.64 \cdot 10^{-20} \text{ J}) / (1.38 \cdot 10^{-23} \text{ J K}^{-1}) = 1190 \text{ K}$$

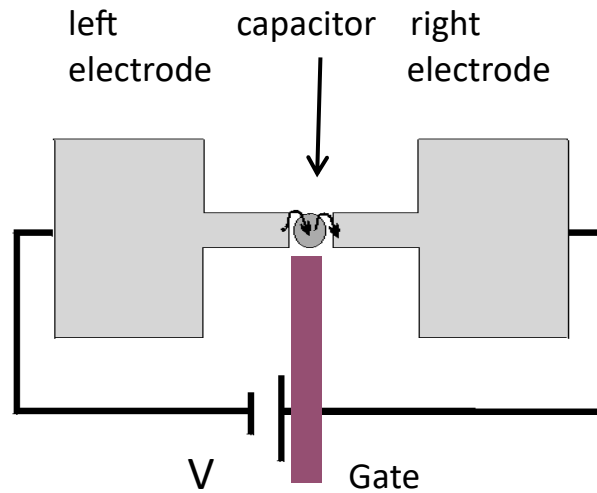
(c) Phonons can be responsible for electron scattering, thus reducing conductivity.

(d) Phonons are responsible for the transitions between the top of the valence band to the bottom of the conduction band: they provide the momentum (not available from photons) that allows electrons to reach the conduction band minimum that, in indirect semiconductors, is not vertically aligned with the valence band maximum.

(e)  $E_0 = (\frac{1}{2}) \hbar\omega$

Zero-Point Energy: it means that, according to quantum mechanics, an harmonic oscillator can never be completely at rest, because then we would know its momentum (zero) and position precisely, which goes against Heisenberg's uncertainty principle. It means that even at absolute zero, the atoms in a material will still be moving around their equilibrium position.

4. (a)



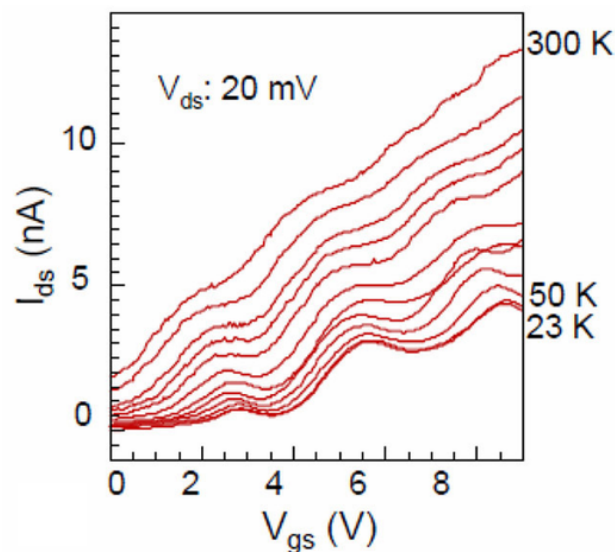
(b) Resolution of the optical microscope will be limited by diffraction, typically about  $\lambda/2$ , so using blue light, the resolution would be around 200nm at best.

(c) Scanning Electron Microscopes use a beam of focused electrons that are accelerated onto the target with kV fields. The wavelength associate with the electrons is much smaller than visible-light wavelength, therefore higher resolutions are achieved, down to 1 nm.

(d)  $C = 10$  aF

Charging energy =  $Q^2/(2C) = (2.6 \cdot 10^{-38} \text{ s}^2 \text{ A}^2)/(20 \cdot 10^{-18} \text{ kg}^{-1} \text{ m}^{-2} \text{ s}^4 \text{ A}^2) = 1.3 \cdot 10^{-21} \text{ J}$

(e)



The single electron charging plateaux will only be visible when the thermal energy is lower than the charging energy. This generally occurs for 10K. Some oscillations are still visible at 100K, while at 300K most of the effect will be “washed out” and the charging steps will not be clearly visible anymore.

(f) Single electron transistors can be used as charge sensor (single electron control), as spin-polarised current sources, as qubits for quantum computers (using the two spin states).