4C15. Dr A Seshia

1 (a) Bookwork but essentially:

- Smooth surfaces
- no friction or adhesion
- linear elasticity
- small strains

(b) (i)

$$P = \frac{4}{3} E^* R^{1/2} (\delta - 2\delta_c)(\delta + \delta_c)^{1/2}$$

if $\delta \gg \delta_c$ then $P = \frac{4}{3} E^* R^{1/2} \delta^{3/2}$

which is data sheet formula rearranged.

But clearly from inspection of given formula $P = 0$ when $\delta = 2\delta_c$ or $\delta = -\delta_c$ so that there is adhesion over range $-\delta_c < \delta < 2\delta_c$

(ii)

$$P = \frac{4}{3} \times 3 \times 10^6 \times (.001)^{1/2} (\delta - 2\delta_c)(\delta + \delta_c)^{1/2}$$

$$\delta_c = \frac{3}{2} (\cdot 001)^{1/2} \left(\frac{80 \times 10^{-3}}{3 \times 10^6}\right)^{2/3} = 1.34 \times 10^{-6} \text{ m}$$

$$P = \frac{4}{3} \times 3 \times 10^6 \times (\cdot 001)^{1/2} (\delta - 2.68 \times 10^{-6})(\delta + 1.34 \times 10^{-6})^{1/2}$$

or $P = 126.5 (\delta - 2.68)(\delta + 1.34)^{1/2}$ μN if δ in μm

whereas for Hertz

$$P_H = 126.5 \, \delta^{3/2}$$

Hence numerical values

-1	δ μm	-1.34	-.5	0	1	1.5	2	2.68	3	4	5	5
-271	P μN	0	-367	-392	-325	-251	-157	0	84	385	739	[illegible]
	P_H μN	–	–	0	127	232	358	555	657	1012	1414	[illegible]

(iii) $P = \frac{4}{3} E^* R^{1/2} (\delta - 2\delta_c)(\delta + \delta_c)^{1/2}$

When $P = 0$ clearly still stable contact with $\delta = 2\delta_c$.

Pull-off will occur when $\frac{dP}{d\delta} = 0$

But $\frac{dP}{d\delta} = \frac{4}{3} E^* R^{1/2} \left\{ (\delta - 2\delta_c) \frac{1}{2} (\delta + \delta_c)^{-1/2} + (\delta + \delta_c)^{1/2} . 1 \right\}$

$= 0$ when $(\delta - 2\delta_c) + 2(\delta + \delta_c) = 0$

i.e. $3\delta = 0$, $\delta = 0$

then $P = \frac{4}{3} E^* R^{1/2} - 2\delta_c \, \delta_c^{1/2}$

So substituting $\delta_c = \frac{3}{2} R^{1/3} \left(\frac{w}{E^*}\right)^{2/3}$

$$P = -\frac{4}{3} E^* R^{1/2} \, 2 \left(\frac{3}{2}\right)^{3/2} R^{1/2} \frac{w}{E^*} = -\frac{8}{3}\left(\frac{3}{2}\right)^{3/2} R w$$

i.e. ~~xxx~~ tensile (as reqd) $P_{pulloff} = 4.9 Rw$, (equivalent to $392 \mu N$)

If this just balances self weight $= \frac{4}{3}\pi R^3 \rho g$

$\frac{4}{3}\pi R^3 \rho g = 4.9 R w$

$R^2 = \frac{4.9 \times 3 \, w}{4\pi \rho g} = \frac{4.9 \times 3 \times 80 \times 10^{-3}}{4\pi \times 1200 \times 9.81}$

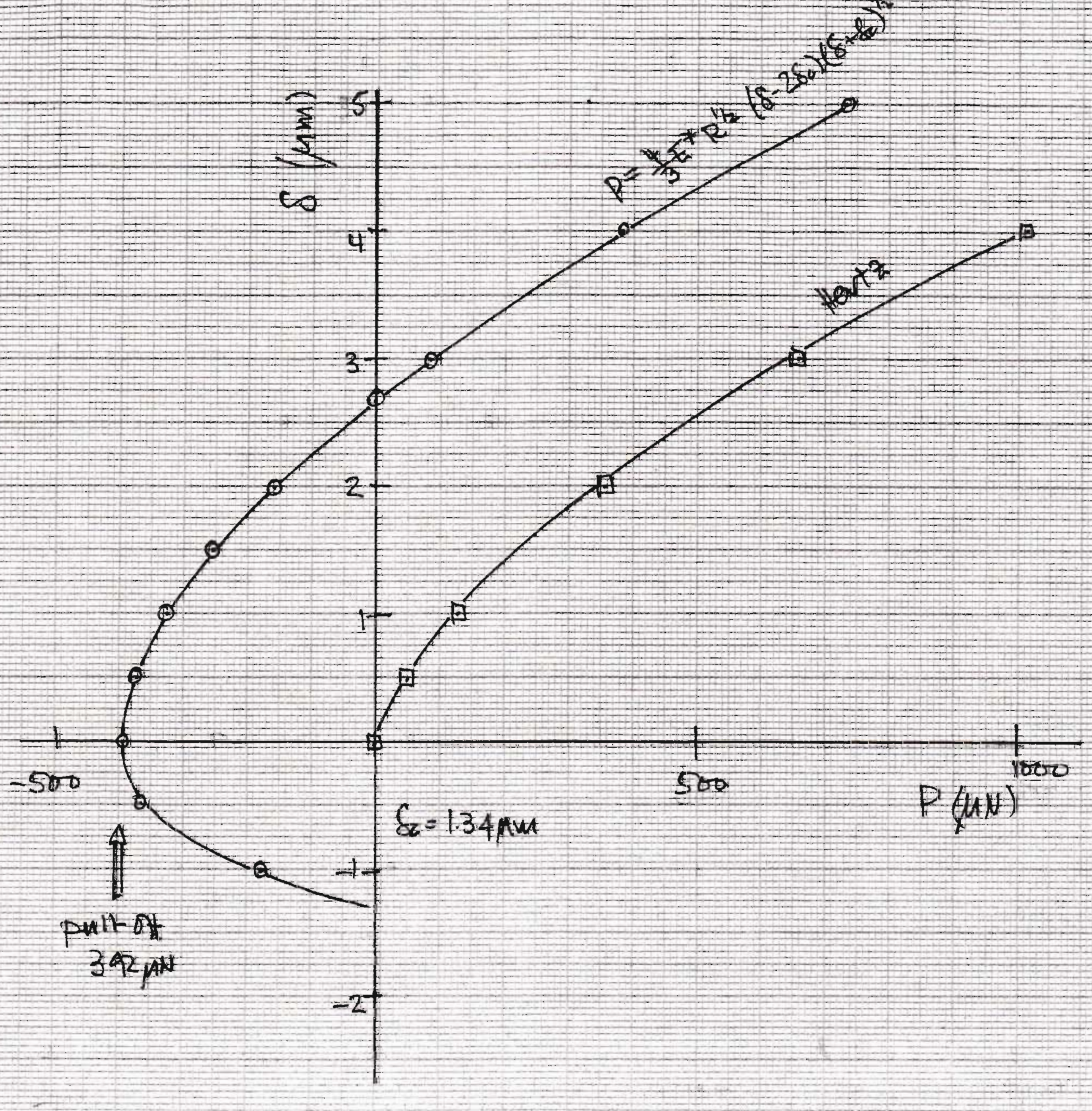
$\therefore R = 2.8 \text{ mm}$

(iv) Dirty slide would reduce w as by reference to values of surface energy in data sheet. So R falls.

A meniscus film might "improve" adhesion. ~~Ex~~ Using eqn from notes i.e. $2\pi R \gamma$ $\gamma = 73.1 \text{ mJm}^{-2}$

$2\pi R\gamma = \frac{4}{3}\pi R^3 \rho g$ so that $R = \sqrt{\frac{3\gamma}{2\rho g}}$

$= \sqrt{3 \times 73.1 \times 10^{-3} / 2 \times 1200 \times 9.81} = 3.1 \text{ mm}$



δ (μm)
$P = \frac{4}{3}E^* R^{1/2} (\delta - 2\delta_c)(\delta + \delta_c)^{1/2}$
Hertz
-500
500
1000
P (μN)
δ_c = 1.34 μm
pull-off
342 μN

Q2

(a)

$$C_{tot} = \text{total capacitance}$$

$$= C_0(x) \,//\, C_1 + C_2$$

$$= \frac{\left(\frac{\epsilon_0 A}{g-x}\right)\frac{\epsilon_0 A}{ng}}{\frac{\epsilon_0 A}{g-x} + \frac{\epsilon_0 A}{ng}} + C_2$$

$$= \frac{\epsilon_0 A}{[(n+1)g - x]} + C_2$$

(b)

$$W = \frac{1}{2} C_{tot} V^2 \quad \text{(stored energy)}$$

$$= \frac{1}{2}\left(\frac{\epsilon_0 A}{[(n+1)g-x]} + C_2\right) V^2$$

$$F_{el} = -\frac{\partial}{\partial x} W \quad \text{(electrostatic force)}$$

$$= -\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\epsilon_0 A}{[(n+1)g-x]} + C_2\right)\right] V^2$$

$$= -\frac{1}{2}\frac{\epsilon_0 A V^2}{[(n+1)g - x]^2}$$

$$F_{tot} = F_{el} + F_{mech}$$

$$= -\frac{1}{2}\frac{\epsilon_0 A V^2}{[(n+1)g-x]^2} + kx$$

$$\frac{\partial F_{tot}}{\partial x} = k - \frac{1}{2}\frac{\epsilon_0 A V^2 (2)}{[(n+1)g - x]^3}$$

at $x = x_{PI}$;

$$k = \frac{\epsilon_0 A V^2}{[(n+1)g - x_{PI}]^3}$$

also $F_{tot} = 0$.

$$\therefore \frac{\epsilon_0 A V^2}{2[(n+1)g - x_{PI}]^2} = \frac{\epsilon_0 A V^2 x_{PI}}{[(n+1)g - x_{PI}]^3}$$

$$\therefore (n+1)g - x_{PI} = 2x_{PI}$$

or

$$x_{PI} = \frac{(n+1)g}{3}$$

as $n \to 0 \quad x_{PI} \to g/3$.

(c) For

$$x_{PI} = g$$

$$\frac{(n+1)g}{3} = g$$

or $n+1 = 3$

$\Rightarrow n = 2$

The capacitor C_1 in series with C_0 must equal half the nominal value of C_0 (or less) to ensure stable operation over the entire gap.

Q5

(a)
$$f = \frac{1}{2\pi}\sqrt{\frac{K_{eff}}{M_{eff}}} = \frac{1}{2\pi}\sqrt{\frac{\pi^2 E h}{\rho L^2 h}} = \frac{1}{2L}\sqrt{\frac{E}{\rho}}$$

$E = 160$ GPa, $\rho = 2330$ kg/m^3

$$f = \frac{1}{2\times 10^{-3}} \times \sqrt{\frac{160\times 10^{9}}{2330}} = 4.14 \text{ MHz}$$

$$f = 1\text{ GHz} \Rightarrow L = \frac{1}{2\times 10^{9}}\sqrt{\frac{160\times 10^{9}}{2330}} = 4.14\ \mu\text{m}$$

Frequency scaling ① Choose materials with high acoustic velocity $\sqrt{E/\rho}$ e.g. diamond, ② Reduce dimension L by lithography. ③ Scale to higher-order modes.

(b)
$$K_{eff} = \pi^2 E h = 15.8 \text{ MN/m}$$

$$M_{eff} = \rho A h = 2.33\times 10^{-8} \text{ kg}$$

$$B_{eff} = \frac{\omega M_{eff}}{Q} = 6.06\times 10^{-6} \text{ N-s/m}$$

$$\eta_{1\text{-port}} = \frac{V_{DC}\,\epsilon_0 A_{el}}{g^2} = \frac{100\times 8.85\times 10^{-12}\times 10^{-8}\times 4}{10^{-12}} = 4\times 8.85\times 10^{-6} \text{ C/V} = 3.54\times 10^{-5} \text{ C/m}$$

$$\eta_{2\text{-port}} = \frac{V_{DC}\,\epsilon_0 A_{el\text{-}2}}{g^2} = \frac{100\times 8.85\times 10^{-12}\times 10^{-8}\times 2}{10^{-12}} = 1.77\times 10^{-5} \text{ C/m}$$

(i) 1-port

$$L_m = M_{eff}/\eta^2 = \frac{2.33\times 10^{-8}}{(3.54\times 10^{-5})^2} = 18.6 \text{ H}$$

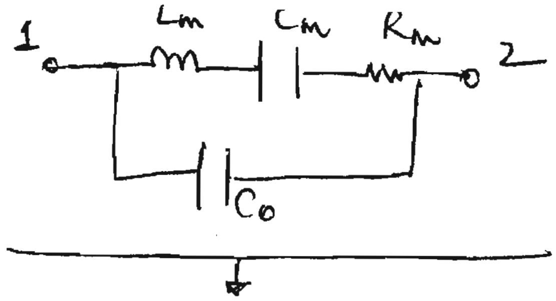
$$C_m = \frac{\eta^2}{K_{eff}} = \frac{(3.54\times 10^{-5})^2}{15.8\times 10^{6}} = 7.93\times 10^{-17} \text{ F}$$

$$R_m = B_{eff}/\eta^2 = \frac{6.06\times 10^{-6}}{(3.54\times 10^{-5})^2} = 4835\ \Omega$$

(ii) 2-port values are given by:

$L_m = 4\times 18.6 = 74.4$ H; $C_m = 1.98\times 10^{-17}$ F; $R_m = 19.34$ kΩ

(c)



$$Y_m(j\omega) = j\omega C_0 + \frac{j\omega C_m}{[(1-\omega/\omega_0)^2] + j\omega/\omega_0 Q}$$ is the expression for admittance

(d) $$f_{res} = \frac{1}{2L}\sqrt{\frac{E}{\rho}}$$

$$\frac{dfres/fres}{dT} \simeq \frac{1}{2}\frac{dE/E}{dT}$$ as $\frac{(dL/L)}{dT}$ is considered small for polysilicon by about an order of magnitude

Approaches to passive temperature compensation include:

1. Degenerate doping
2. Coating the resonator with a material describing opposite dependance of E on T e.g SiO_2 or embedding SiO_2 pillars in polysilicon
3. Temperature compensation by inducing mechanical stress in the resonator in a well-defined manner as a function of temp to compensate for inherent dependance.

(4) (a)

$$\Delta C = \frac{C_s}{g}\left[\frac{g}{g-x} - \frac{g}{g+x}\right].$$

$$\approx C_s \cdot \frac{2xg}{g^2}$$

$$= C_s\left(\frac{2x}{g}\right)$$

$$x = \frac{ma}{k} = \frac{a}{\omega_r^2}$$

$$\Delta C / C_s = 2\left(\frac{a}{\omega_r^2}\right)\cdot\frac{1}{g}$$

(b)

$$\overline{x_n^2} = \frac{4k_B T b}{k^2}$$

$$\overline{a_n^2} = \frac{k^2}{m^2}\overline{x_n^2} = \frac{4k_B T b}{m^2}$$

min. det. acceleration in 1Hz BW $\approx 2\frac{\sqrt{4k_B T b}}{m} \approx 4\frac{\sqrt{k_B T b}}{m}$

(c)

$$\overline{a_n^2} = \frac{4k_B T b}{m^2} = \frac{4k_B T \omega_0 m}{m^2 Q} = \frac{4k_B T \omega_0}{mQ}$$

min. detectable acceleration $= 4\sqrt{\frac{k_B T \omega_0}{mQ}}$

$$= 4\sqrt{\frac{1.38\times10^{-23}\times300\times2\pi\times10^3}{10^{-6}\times10}}$$

$$= 6.45\ \mu^{m/s^2}/\sqrt{Hz}$$

(d)

$$\left(\frac{\Delta C}{C_s}\right) = 2\left(\frac{a}{\omega_r^2}\right)\frac{1}{g}$$

$$\therefore a_{min} = \left(\frac{\Delta C}{C_s}\right)_{min}\cdot\frac{\omega_r^2 g}{2} = \frac{10^{-6}\times(2\pi\times10^3)^2\times10^{-6}}{2} = 19.7\ \mu m/s^2$$

(e) Design approaches to reduces minimum detectable noise floor include -

- better resolution of capacitance changes
  reduce noise of interface circuit for capacitance measurement
- reduce natural frequency (↑ sensitivity)
- decrease capacitive gap (if electronic noise limits)
- increase proof mass (larger area or more complex process)
- reduce damping (e.g. by vacuum packaging)
- reduce temperature (e.g. by active cooling)