EGT3
ENGINEERING TRIPOS PART IIB

Monday 2 May 2022   9.30 to 11.10

Module 4C2

DESIGNING WITH COMPOSITES

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C2 Designing with Composites data sheet (6 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Describe briefly what is meant by a *laminate* and explain why they are commonly employed in preference to a simple unidirectional lamina. Explain what is meant by (i) a *balanced laminate* and (ii) a *symmetric laminate* when referring to the laminate's elastic response. [15%]

(b) A $[0/\pm 45/0]_s$ laminate is made from eight laminae, each of thickness 0.5 mm. Each lamina consists of 50 vol% Kevlar fibres (E_f = 76 GPa) in an epoxy resin matrix (E_m = 3 GPa). Calculate the laminate extensional stiffness $[A]$ with respect to the global axes (x, y), where x is parallel to the 0° laminae. Comment on its form. [Additional lamina material properties: G_{12} = 2.4 GPa, ν_{12} = 0.3.] [35%]

(c) The laminate in part (b) is subjected to a tensile load along the x direction, which gives rise to an axial strain ε_x.

(i) Find the transverse strain ε_y in terms of ε_x. [20%]

(ii) Calculate the in-plane stresses for each lamina in the principal axes (1, 2) in terms of ε_x. [30%]

2 (a) Discuss, with illustrative examples, why there is such a wide range of fibre architectures used in engineering composites. [20%]

(b) Consider a $[\pm 45/0]_s$ laminate made from six laminae, each 0.2 mm thick, of AS/3501 CFRP (material properties on the datasheet). A line load N_x is applied along the 0° direction. The stiffness matrices $[Q]_0$ and $[\overline{Q}]_{45}$ in the laminate global axes for the 0° and 45° laminae, respectively, are as follows:

$$[Q]_0 = \begin{bmatrix} 139 & 2.7 & 0 \\ 2.7 & 9 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{GPa}, \qquad [\overline{Q}]_{45} = \begin{bmatrix} 45 & 31 & 32 \\ 31 & 45 & 32 \\ 32 & 32 & 36 \end{bmatrix} \text{GPa}$$

(i) Find the line load N_x at failure, using the Tsai-Hill failure criterion and assuming that failure occurs in one of the ±45° plies. Identify the corresponding failure mode. [55%]

(ii) It is proposed to increase the strength of the laminate by choosing a higher modulus carbon fibre for the 0° plies, without any change in the fibre strength for these plies, or any change in material for the ±45° plies. Explain how this approach could increase the laminate strength. Estimate, without repeating the detailed laminate calculations of part (b)(i), the approximate change in laminate strength that would result from a 50% increase in the fibre stiffness. [25%]

3 (a) Discuss how the performance requirements of an ice hockey stick are likely to affect the way in which composite materials can be used in its design. [20%]

(b) Explain the concept of the testing pyramid as used for composite aerospace applications, identifying strengths and weaknesses in the approach. How would you modify this approach for testing an ice-hockey stick? [20%]

(c) Figure 1(a) shows a tubular structure of length L. The tube has a hollow circular cross section, as illustrated in Fig. 1(b), with radius R and wall thickness t. It can be assumed that t is much less than R, so that the second moment of area of the tube can be approximated by $\pi t R^3$. The tube is fixed at one end. At the free end, a bending moment M and torsional load Q are applied. The tube wall is made up from CFRP laminae orientated at 0°, 90° and ±45° to the axial direction of the tube. Figure 2 contains carpet plots for this material. The tube contains a minimum of 10% of 90° plies.

(i) Consider failure of the tube due to a combination of the bending moment M and torque Q. Use data from Fig. 2 and from the datasheet to sketch the proportion of 0° plies needed to minimise the beam mass, while avoiding failure, as a function of the ratio of the bending to torsional loads, M/Q. [40%]

(ii) Carbon fibres are replaced by glass fibres in the ±45° plies, but not in the 0° and 90° plies. Without making any additional calculations, explain qualitatively how this will change your answer for (c)(i). [20%]

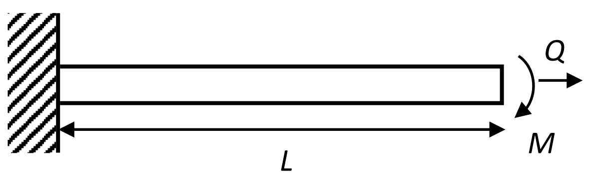


Fig. 1(a)

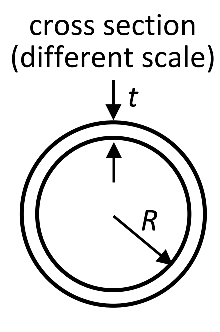


Fig. 1(b)

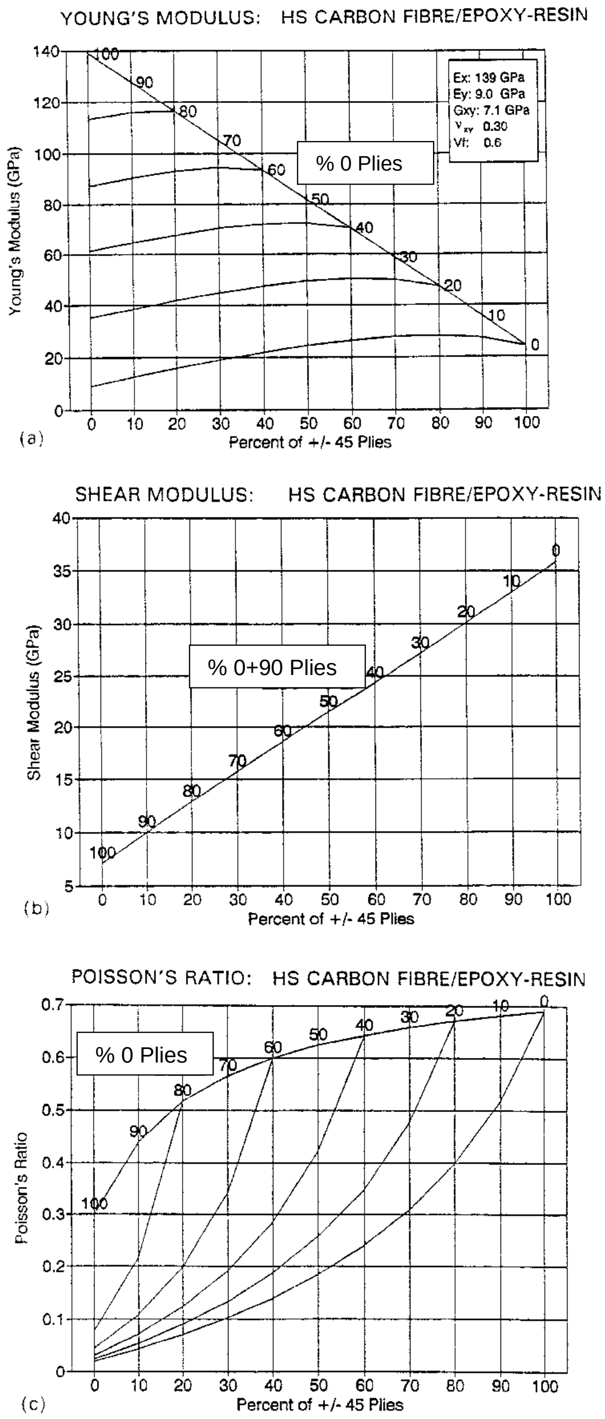

YOUNG'S MODULUS: HS CARBON FIBRE/EPOXY-RESIN
Ex: 139 GPa
Ey: 9.0 GPa
Gxy: 7.1 GPa
ν_{xy} 0.30
Vf: 0.6
% 0 Plies
Young's Modulus (GPa)
Percent of +/- 45 Plies
(a)
SHEAR MODULUS: HS CARBON FIBRE/EPOXY-RESIN
% 0+90 Plies
Shear Modulus (GPa)
Percent of +/- 45 Plies
(b)
POISSON'S RATIO: HS CARBON FIBRE/EPOXY-RESIN
% 0 Plies
Poisson's Ratio
Percent of +/- 45 Plies
(c)

Fig. 2

4 (a) Discuss the role that the bond between fibre and matrix plays in the mechanical properties of fibre composites. [20%]

(b) Figure 3(a) illustrates a test to measure the bond between a metal fibre and a block of ceramic matrix. The fibre has a circular cross section with diameter d and behaves elastically with a Young's modulus E. It is initially embedded a distance L into a large block of matrix, assumed rigid. An increasing pull-out load P is applied to the fibre a distance L above the matrix block, with a corresponding displacement Δ, until the fibre is entirely pulled out from the matrix. The bond between the fibre and matrix is modelled in an elastic-plastic manner, with the shear stress τ at the interface in the embedded section depending on the local relative displacement δ between the fibre and the matrix as shown in Fig. 3(b). Plastic slip between the fibre and the matrix occurs when the shear stress and displacement exceed critical values of τ_f and δ_f, respectively. Below this critical value, the shear stress depends linearly on the displacement. Make appropriate simplifying assumptions in answering the following questions, detailing the assumptions you make.

(i) Sketch the expected relationship between the applied load P and the remote displacement Δ, indicating on the sketch the corresponding mechanisms of behaviour, but without at this stage doing any detailed calculations. [20%]

(ii) Derive an expression for the maximum value of the pull-out load, in terms of L, d and τ_f. [15%]

(iii) Derive an approximate expression for the plastic work done due to pull-out. [20%]

(iv) Derive an expression for the load P at which plastic slip between the fibre and matrix first occurs. [25%]

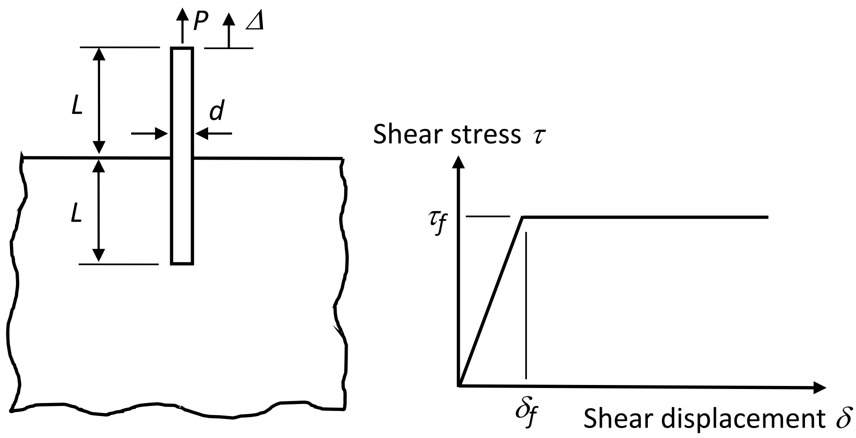


Fig. 3(a) Fig. 3(b)

END OF PAPER

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = [S] \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} \qquad \text{where } [S] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}$$

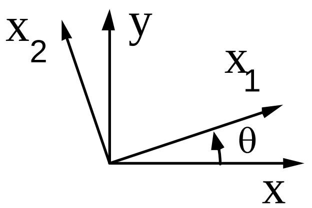
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}$$

where
$$\begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ , with respect to the (x, y) axes.



Then,
$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = [T]^{-T} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

where
$$[\mathrm{T}] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

and
$$[\mathrm{T}]^{-T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)s^2c^2 + Q_{12}\left(c^4 + s^4\right) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)sc^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s^3c \\ \overline{Q}_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)s^3c - \left(Q_{22} - Q_{12} - 2Q_{66}\right)sc^3 \\ \overline{Q}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)s^2c^2 + Q_{66}\left(s^4 + c^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

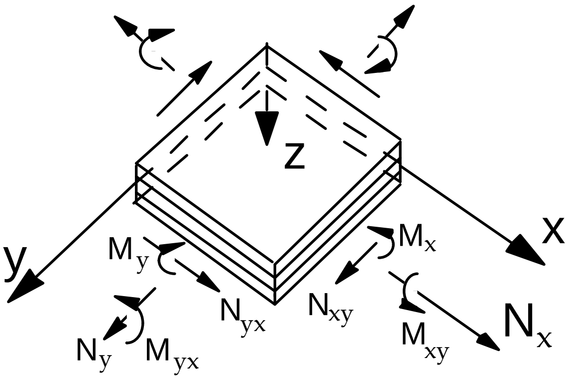
The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}c^4 + S_{22}s^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{12} &= S_{12}\left(c^4 + s^4\right) + \left(S_{11} + S_{22} - S_{66}\right)s^2c^2 \\ \overline{S}_{22} &= S_{11}s^4 + S_{22}c^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{16} &= \left(2S_{11} - 2S_{12} - S_{66}\right)sc^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)s^3c \\ \overline{S}_{26} &= \left(2S_{11} - 2S_{12} - S_{66}\right)s^3c - \left(2S_{22} - 2S_{12} - S_{66}\right)sc^3 \\ \overline{S}_{66} &= \left(4S_{11} + 4S_{22} - 8S_{12} - 2S_{66}\right)s^2c^2 + S_{66}\left(c^4 + s^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts i, j = 1, 2 or 6.

Here,

n = number of laminae

t = laminate thickness

$z_{k\text{-}1}$ = distance from middle surface to the top surface of the k-*th* lamina

z_k = distance from middle surface to the bottom surface of the k-*th* lamina

<u>Quadratic failure criteria.</u>

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $F_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

MPF Sutcliffe
NA Fleck
AE Markaki