EGT3
ENGINEERING TRIPOS PART IIB

Monday 1 May 2023   9.30 to 11.10

Module 4C2

DESIGNING WITH COMPOSITES

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C2 Designing with Composites data sheet (6 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) How many independent elastic constants are required to define the stress-strain relationship for a thin unidirectional composite lamina under plane stress conditions? Describe how you can experimentally measure its engineering constants. Use diagrams to illustrate your answer. [25%]

(b) The wall of a hollow cylindrical drive shaft is fabricated from a laminate made of AS/3501 carbon fibre epoxy material (material properties given on the datasheet). The stacking sequence is $[(-60/0/+60)_4]_s$ with respect to the longitudinal axis of the shaft, as shown in Fig. 1. Each ply has a thickness of 0.1 mm. The shaft approximates to a thin-walled cylinder with an outer radius of 45 mm. During service, the shaft must sustain an axial tensile load P_x = 50 kN and a torque T_{xy} = 10 kN m.

(i) Calculate the extensional stiffness matrix $[A]$ for the wall material in axes (x, y) aligned with those of the shaft. Comment on its form. [30%]

(ii) Calculate the strains ε_x, ε_y, γ_{xy} in the shaft due to the combined tension and torsion. [25%]

(iii) Discuss how in practice you would implement this drive shaft solution, including manufacturing considerations and layup optimisation. [20%]

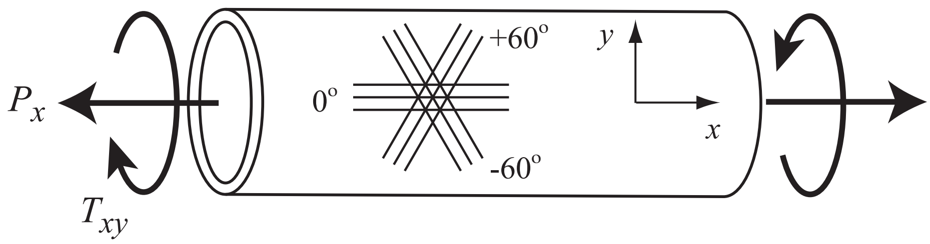


Fig. 1

2 (a) Discuss how you would measure the in-plane mechanical properties of composites, including the role of testing in design. Illustrate three methods of critical testing. [25%]

(b) Consider a $[(0_2, 90)_n]_s$ laminate containing $6n$ plies, each 0.125 mm thick, of AS/3501 CFRP (material properties given on the datasheet). The laminate is subject to a biaxial stress state, with tensile line loads of 1.50 MN m^{-1} and 0.50 MN m^{-1} acting along and perpendicular to the 0° direction, respectively. The stiffness matrix $[Q]$ for 0° laminae is

$$[Q] = \begin{bmatrix} 139 & 2.7 & 0 \\ 2.7 & 9.0 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{GPa.}$$

(i) Find the minimum value for the number of plies (equal to $6n$) needed to carry the biaxial loads, using a maximum stress failure criterion. [55%]

(ii) Using approximate calculations, estimate the proportion of 0° and 90° plies (i.e. not restricted by the ply blocking arrangement given above) which would minimise the thickness of the laminate required to carry the loads. What is the corresponding laminate thickness? [20%]

3 (a) Discuss the role of cracking in composite design. Comment specifically on testing, design to avoid cracking, and modelling of cracking failure. [20%]

(b) Consider a $[\pm 45, 0]_s$ laminate made from 6 plies of CFRP. The laminate contains a sharp notch of length 50 mm aligned with the +45° direction, and is subject to a remote tensile stress σ acting in the 0° direction. The laminate in-plane dimensions are much larger than the notch length. The laminate compliance matrix $[S]$ is given by

$$\begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} 0.020 & -0.010 & 0 \\ -0.010 & 0.040 & 0 \\ 0 & 0 & 0.050 \end{bmatrix} \text{GPa}^{-1}.$$

The mode I and mode II laminate toughnesses are $G_{IC} = 20$ kJ m^{-2} and $G_{IIC} = 50$ kJ m^{-2}. Estimate the stress σ associated with failure at the notch tip, detailing any assumptions that you make about the effect of mode mixity on toughness. [50%]

(c) Explain the following observations.

(i) Through thickness cracking is particularly important to consider in thick composites. [10%]

(ii) Composite joints are susceptible to interlaminar fatigue cracking. [10%]

(iii) Uniformity of fibre strengths within a long fibre composite laminate is important in determining its axial tensile strength. [10%]

4 You have been asked to re-design the structure of an autonomous grocery delivery vehicle, essentially a cubical box of side length 800 mm on wheels navigating the streets and pavements of Cambridge.

(a) Discuss the design and manufacture of such a structure using composites, including comments on material and layup, structural design and manufacturing (but do not limit yourself to just these points). [30%]

(b) Figure 2 illustrates a critical load case for the top panel of the vehicle. The panel is a sandwich structure of side length L = 800 mm and total thickness c = 10 mm. Face sheets each of thickness t are separated by a core material. The weight and potential failure of the core material can be neglected. The panel is loaded uniformly along the centreline by a total load of P = 4 kN and is simply supported at the edges. Both orientations for the load need to be considered, as illustrated in Fig. 2. In that figure, the panel orientation is unchanged but the load orientation and which edges are assumed to support the load do change for the two load cases. Use approximate calculations and the data in Table 1 of the datasheet to select an appropriate composite material, face sheet thickness t and layup for the following design cases.

(i) The weight of the panel should be minimised whilst avoiding failure of the face sheets. [35%]

(ii) The profit of the panel should be maximised whilst avoiding failure of the face sheets, including a premium of £200 per kilogram of weight saved. [15%]

(iii) As per case (ii), but also ensuring that the central deflection of the panel is less than 10 mm. [20%]

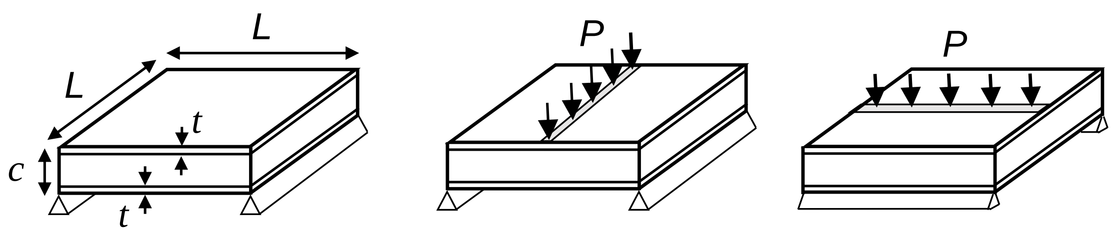


Fig. 2

END OF PAPER

THIS PAGE IS BLANK

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[S]\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix} \qquad \text{where } [S]=\begin{bmatrix}1/E_1 & -\nu_{21}/E_2 & 0\\ -\nu_{12}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

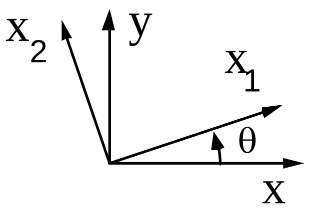
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}$$

where
$$\begin{aligned}Q_{11} &= E_1/(1-\nu_{12}\nu_{21})\\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21})\\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{66} &= G_{12}\end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ, with respect to the (x, y) axes.



Then,
$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=[T]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[T]^{-T}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$$

where
$$[\mathrm{T}]=\begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

and
$$[\mathrm{T}]^{-T}=\begin{bmatrix}\cos^2\theta & \sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)\end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)s^2c^2 + Q_{12}\left(c^4 + s^4\right) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)sc^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s^3c \\ \overline{Q}_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)s^3c - \left(Q_{22} - Q_{12} - 2Q_{66}\right)sc^3 \\ \overline{Q}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)s^2c^2 + Q_{66}\left(s^4 + c^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

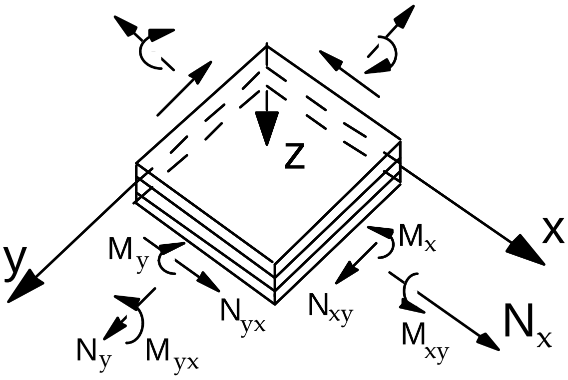
The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}c^4 + S_{22}s^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{12} &= S_{12}\left(c^4 + s^4\right) + \left(S_{11} + S_{22} - S_{66}\right)s^2c^2 \\ \overline{S}_{22} &= S_{11}s^4 + S_{22}c^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{16} &= \left(2S_{11} - 2S_{12} - S_{66}\right)sc^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)s^3c \\ \overline{S}_{26} &= \left(2S_{11} - 2S_{12} - S_{66}\right)s^3c - \left(2S_{22} - 2S_{12} - S_{66}\right)sc^3 \\ \overline{S}_{66} &= \left(4S_{11} + 4S_{22} - 8S_{12} - 2S_{66}\right)s^2c^2 + S_{66}\left(c^4 + s^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o,\varepsilon_y^o,\varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x,\kappa_y,\kappa_{xy}\right)^T$. The stress resultants $\left(N_x,N_y,N_{xy}\right)^T$ and bending moment per unit length $\left(M_x,M_y,M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts i, j = 1, 2 or 6.

Here,

n = number of laminae

t = laminate thickness

$z_{k\text{-}1}$ = distance from middle surface to the top surface of the k-*th* lamina

z_k = distance from middle surface to the bottom surface of the k-*th* lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $\mathrm{F}_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2 ; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2 ; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

MPF Sutcliffe
NA Fleck
AE Markaki