EGT3
ENGINEERING TRIPOS PART IIB

Monday 3 May 2021   9.00 to 10.40

Module 4C2

DESIGNING WITH COMPOSITES

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 4C2 Designing with Composites data sheet (6 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) Discuss the factors affecting the stiffness of engineering composite materials, illustrating your discussion with practical examples of different materials. [40%]

(b) A unidirectional lamina consists of 65 vol% glass fibres (E_f = 76 GPa, ν_f = 0.22) in an epoxy resin matrix (E_m = 2.4 GPa, ν_m = 0.34).

(i) Show that the lamina elastic properties can be estimated from simple models as: E_1 = 50.2 GPa, E_2 = 6.48 GPa and G_{12} = 2.43 GPa. Comment on the expected accuracy of these estimates. [25%]

(ii) Calculate the stiffness matrix $[Q]$ of the lamina. The lamina Poisson's ratio ν_{12} can be taken as 0.262. [10%]

(c) A flat laminate with stacking sequence $[\pm 45/90]_s$ is manufactured from six plies, each of 0.5 mm thickness, of a material with the following stiffness matrix $[Q]$ for a unidirectional laminate:

$$[Q] = \begin{bmatrix} 55 & 1.5 & 0 \\ 1.5 & 6.0 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \text{GPa}$$

Calculate the stiffness matrix $[A]$ for the laminate with respect to the global axes (x, y), where x is aligned with the nominal 0° direction of the laminate. [25%]

2 (a) Discuss the merits of different moulding processes for manufacturing composite parts, using example applications to illustrate the points you are making. You do **not** need to describe the different processes. [30%]

(b) Two laminates, a unidirectional $[0_3]$ laminate and a crossply $[0/90/0]$ laminate, are made from AS/3501 carbon/epoxy plies (material properties on the data sheet). The stiffness matrix $[Q]$ for a unidirectional laminate of the material is as follow:

$$[Q] = \begin{bmatrix} 139 & 2.7 & 0 \\ 2.7 & 9.0 & 0 \\ 0 & 0 & 6.9 \end{bmatrix} \text{GPa}$$

The laminates are subject to biaxial loading σ_1 and σ_2 along and transverse to the nominal laminate 0° direction, respectively. Calculate the failure stress for equal applied compressive stresses ($\sigma_1 = \sigma_2$) using the Tsai-Hill failure criterion for:

(i) the unidirectional $[0_3]$ laminate; [20%]

(ii) the crossply $[0/90/0]$ laminate. [50%]

3 (a) Discuss factors which affect the tensile failure of long-fibre reinforced plastic composites. Include sketches of relevant failure processes. [25%]

(b) What materials are likely to be used in a high-performance aeroplane propeller blade? Explain the reasons dictating your suggested materials. [20%]

(c) What is meant by 'ply drops' in composite laminates? Why do they occur and what issues arise at ply drops? [15%]

(d) It is observed that the arches of a baby gym, made of a composite rod, have failed in transit from the manufacturer to the retailer. Explain why this might be. How could this be avoided? [15%]

(e) You are asked to design a composite mast for wind-surfing. Discuss the factors that you would need to consider, including the following points: material and layup, manufacturing route, structural design and any other relevant aspects.

[25%]

4 A thin-walled CFRP tube of circular cross section, with radius $R = 0.1$ m and length $L = 2.8$ m, is used as a drive shaft in an aerospace application.

(a) The critical frequency f (in Hertz) for whirl of the simply-supported drive shaft is given by the expression

$$f = \frac{\pi R}{2L^2}\sqrt{\frac{E}{2\rho}}$$

where ρ is the material density. Briefly outline how this expression is derived and explain what laminate stiffness should be taken as the relevant tube elastic modulus E. Why is the critical frequency independent of tube wall thickness t? [10%]

(b) The tube wall is made up from sixteen CFRP lamina each of thickness 0.125 mm, orientated at 0, 90 and ±45° to the axial direction of the tube. Use the carpet plots of Fig. 1 and any relevant approximate design data from the data sheet to identify a suitable ply lay-up which maximises the torque Q which the drive shaft can carry, while keeping the critical frequency f above 80 Hz. [25%]

(c) For the laminate you have selected in part (b), estimate the torque Q at failure using:

(i) the laminate strain allowable data in Table 1 of the data sheet; [20%]

(ii) lamina maximum strain criteria of $e_L^+ = 1.1\%$, $e_L^- = 0.8\%$, $e_T^+ = 0.5\%$, $e_T^- = 2.0\%$ and $e_{LT} = 2.0\%$. [20%]

(d) Discuss practical testing which would be needed to support the design of the drive shaft, including sketches as appropriate. [25%]

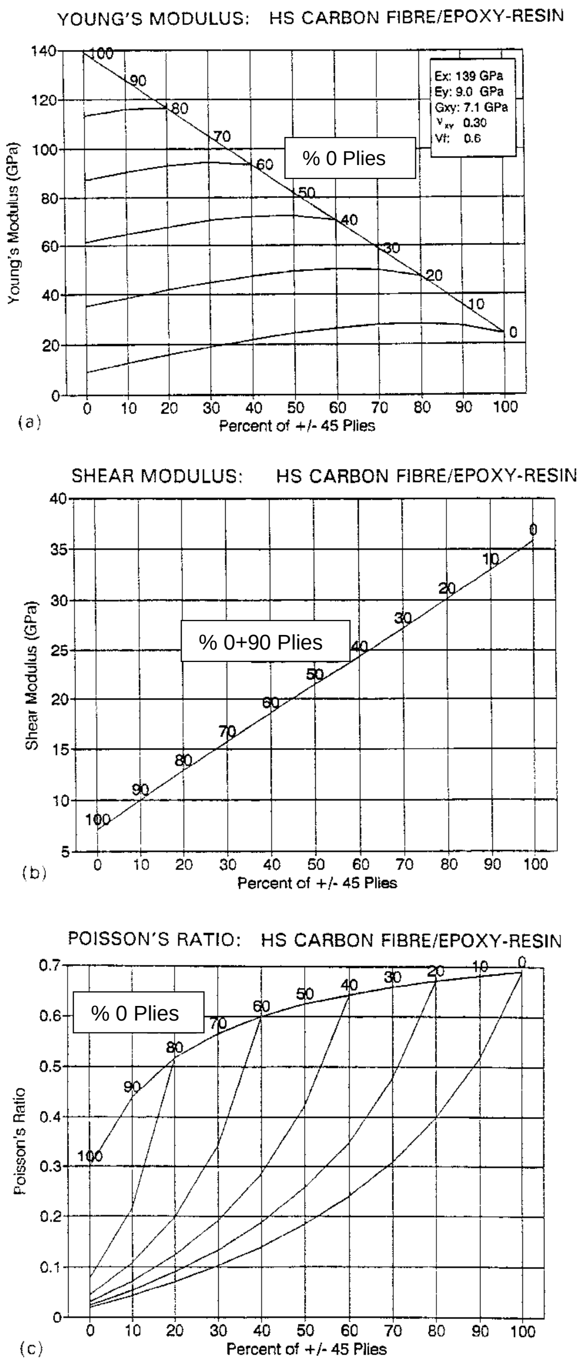

Fig. 1

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ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

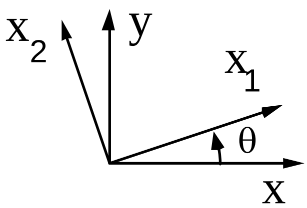
$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[S]\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix} \qquad \text{where } [S]=\begin{bmatrix}1/E_1 & -\nu_{21}/E_2 & 0\\ -\nu_{12}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix} \qquad \begin{aligned}\text{where } Q_{11} &= E_1/(1-\nu_{12}\nu_{21})\\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21})\\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{66} &= G_{12}\end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ , with respect to the (x, y) axes.



Then, $$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=[T]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[T]^{-T}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$$

where $$[\mathrm{T}]=\begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

and $$[\mathrm{T}]^{-T}=\begin{bmatrix}\cos^2\theta & \sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)\end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \text{ where } \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^4) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sc^3 - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})sc^3 \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

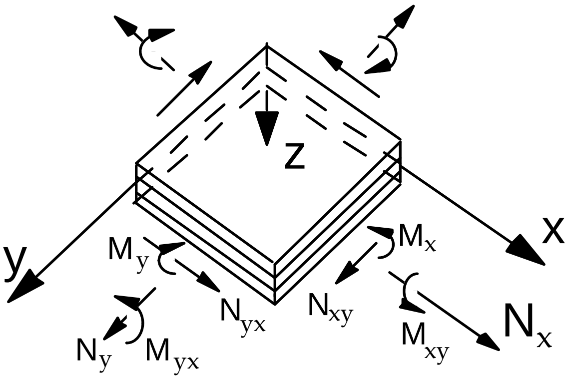
The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}c^4 + S_{22}s^4 + (2S_{12} + S_{66})s^2c^2 \\ \overline{S}_{12} &= S_{12}(c^4 + s^4) + (S_{11} + S_{22} - S_{66})s^2c^2 \\ \overline{S}_{22} &= S_{11}s^4 + S_{22}c^4 + (2S_{12} + S_{66})s^2c^2 \\ \overline{S}_{16} &= (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c \\ \overline{S}_{26} &= (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3 \\ \overline{S}_{66} &= (4S_{11} + 4S_{22} - 8S_{12} - 2S_{66})s^2c^2 + S_{66}(c^4 + s^4) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts i, j = 1, 2 or 6.

Here,

n = number of laminae

t = laminate thickness

$z_{k\text{-}1}$ = distance from middle surface to the top surface of the *k-th* lamina

z_k = distance from middle surface to the bottom surface of the *k-th* lamina

<u>Quadratic failure criteria.</u>

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $F_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

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N. A. Fleck
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October 2008

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

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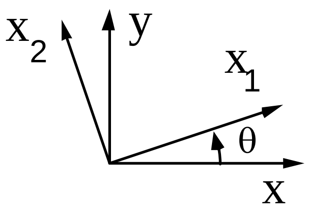
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}$$

$$\text{where } \begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ, with respect to the (x, y) axes.



Then, $\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$ and $\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = [T]^{-T} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$

$$\text{where } [\mathrm{T}] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$\text{and } [\mathrm{T}]^{-T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix}$$

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In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \text{ where } \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}\left(c^4 + s^4\right) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sc^3 - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})sc^3 \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}\left(s^4 + c^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

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and in component form,

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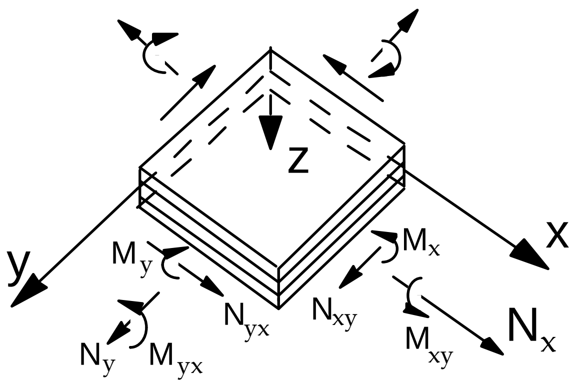
Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts i, j = 1, 2 or 6.

Here,

n = number of laminae

t = laminate thickness

$z_{k\text{-}1}$ = distance from middle surface to the top surface of the *k-th* lamina

z_k = distance from middle surface to the bottom surface of the *k-th* lamina

<u>Quadratic failure criteria.</u>

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\frac{\sigma_1^2}{s_L^2} - \frac{\sigma_1\sigma_2}{s_L^2} + \frac{\sigma_2^2}{s_T^2} + \frac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \frac{1}{s_L^+ s_L^-}$, $F_{22} = \frac{1}{s_T^+ s_T^-}$, $F_1 = \frac{1}{s_L^+} - \frac{1}{s_L^-}$, $F_2 = \frac{1}{s_T^+} - \frac{1}{s_T^-}$, $F_{66} = \frac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2 ; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2 ; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

M. P. F. Sutcliffe
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Engineering Tripos Part IIB: Module 4C2
Designing with Composites

Numerical answers – 2020/21

1. (b) $[Q] = \begin{bmatrix} 50.7 & 1.71 & 0 \\ 1.71 & 6.54 & 0 \\ 0 & 0 & 2.43 \end{bmatrix}$ GPa

1. (c) $[A] = \begin{bmatrix} 43 & 28.5 & 0 \\ 28.5 & 92 & 0 \\ 0 & 0 & 31.5 \end{bmatrix}$ MNm^{-1}

2. (b)(i) 248 MPa. (ii) 447 MPa

3. (c)(i) 17.9 kN m (ii) 35.8 kN m (for a laminate with no 90s, four 0s and twelve 45s).