Version JPT/4

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 25 April 2023 2 to 3.40

Module 4C6

ADVANCED LINEAR VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (9 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Figure 1 shows a simple model of a three-storey building. The building is steel-framed with bolted joints. Each floor of the building has mass m, and the lateral stiffness between floors is k, i.e. the restoring force between a pair of floors is k times their relative displacement. The lateral deflection of the n^{th} floor with respect to ground is denoted y_n.

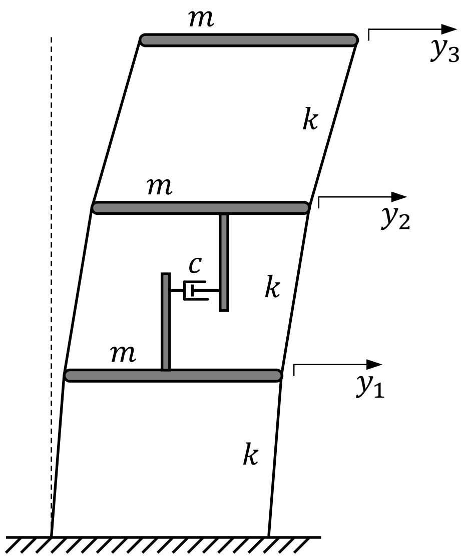

Fig. 1

(a) The *undamped* natural frequencies ω_n have been calculated numerically, giving:

$$\frac{\omega_n^2 m}{k} = \begin{bmatrix} 0.1981 & 1.5550 & 3.2470 \end{bmatrix}$$

Find the corresponding *undamped* mode shapes. [10%]

(b) List three mechanisms of damping that you would expect to be significant for this structure and justify your answers. [10%]

(c) In order to increase the damping of the structure, a viscous damper with dashpot rate c is now added between the first and second floors as shown in Fig. 1. Write down an expression for the Rayleigh dissipation function, and find the damping matrix. [10%]

(d) Write down the unforced equation of motion for the structure, including the damper, in the form:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$$

Write out your definition of the full vector $\mathbf{z}$ and all entries of the matrix $\mathbf{A}$. [20%]

 (cont.

(e) The eigenvalue vector '`lambda_A`' and eigenvector matrix '`Phi_A`' of $\mathbf{A}$ have been computed numerically, using the following values: $m = 1\,\text{kg}$, $k = 100\,\text{Nm}^{-1}$, $c = 1\,\text{Nsm}^{-1}$ (representing a laboratory scale model of the building). Three of the numerically computed eigenvalues and corresponding eigenvectors are reported as:

```
lambda_A =

 [  -0.883 + 17.979j    -0.083 + 12.48j     -0.035 + 4.451j ]


Phi_A =

 [[ -0.019 + 1.j         0.007 + 1.j        -0.01  + 1.j     ]
  [ -0.06  - 1.231j     -0.045 + 0.449j      0.021 + 1.801j ]
  [  0.104 + 0.538j      0.051 - 0.807j      0.017 + 2.246j ]
  [-17.962 - 1.229j    -12.48  - 0.j        -4.451 - 0.079j ]
  [ 22.179 - 0.j        -5.594 - 0.603j     -8.017 + 0.031j ]
  [ -9.771 + 1.394j     10.062 + 0.709j     -9.998 - 0.j     ]]
```

(i) What is the physical interpretation of each eigenvector having a length of 6? [5%]

(ii) What is the physical interpretation of the mode shapes being complex? [10%]

(iii) How many eigenvalues and eigenvectors are there in total, and how do the missing ones relate to the ones printed above? [10%]

(iv) Make a table that shows the natural frequency, the mode shape, and the Q-factor for each mode. [15%]

(v) How would you expect the results to change if a damper was included between each floor (including between ground and the first floor)? [10%]

2 The out-of-plane displacement y of a circular membrane with uniform tension P and mass per unit area ρ is expressed in polar coordinates (r, θ), where θ is measured in radians. The partial differential equation that describes $y(r, \theta, t)$ for the membrane is given by:

$$P\left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2}\right) = \rho\frac{\partial^2 y}{\partial t^2}$$

The membrane is fixed at a radius $a = 0.4\,\mathrm{m}$ and the wave speed is $\sqrt{P/\rho} = 40\,\mathrm{ms}^{-1}$. For reference Fig. 2 shows the first four Bessel functions of the first kind, labelled with the zero-crossings of each function.

(a) Using the separation of variables method, with $y = R(r)F(\theta)\mathrm{e}^{i\omega t}$, find the differential equations satisfied by $R(r)$ and $F(\theta)$. [20%]

(b) Sketch the nodal lines for all modes with up to two nodal diameters or two nodal circles (not counting the boundary) or both. [20%]

(c) In order of increasing frequency, list the first six modes of the membrane. Label the modes as (m, n), where m is the number of nodal circles, and n is the number of nodal diameters, and find numerical values for the natural frequencies. [20%]

(d) An accelerometer with a small (but non-negligible) mass is placed at coordinates $(0.435a, 0)$. The membrane is struck with an instrumented hammer at $(0.9a, \pi/2)$. Sketch the transfer function that you would expect on a dB scale, over the range of frequencies spanned by the modes listed in part (c), labelling key frequencies. [30%]

(e) Describe briefly what happens to the natural frequencies of the membrane as the added mass of the accelerometer increases, including what would happen as the mass tends to infinity. [5%]

(f) How would the behaviour referred to in part (e) change if the mass was removed, and instead the membrane was connected to ground by a spring of infinite stiffness? [5%]

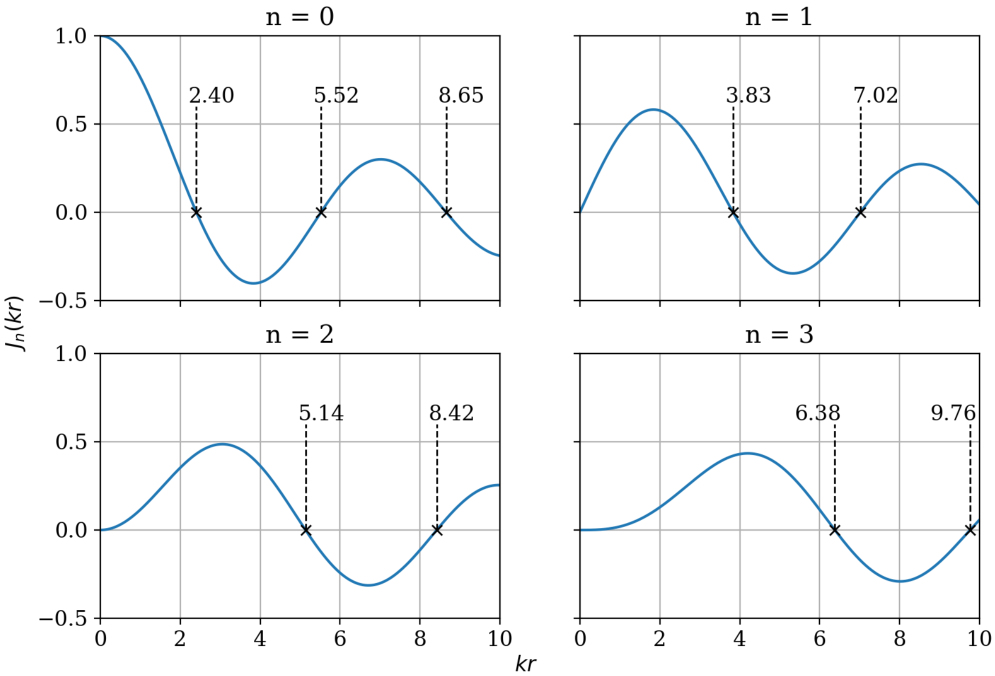

Fig. 2

(TURN OVER

3 An electromagnetic shaker and LVDT displacement transducer are used to measure the response of a cantilevered floor beam by conducting a frequency sweep over the first three modes of vibration. Table 1 summarises measurements of the driving-point response at the tip of the cantilever, for each mode, together with the pseudo-static response at 0.1 Hz.

Mode	Frequency (Hz)	Response amplitude (mm/N)
-	0.1	1.13
1	9.6	41.56
2	60.3	1.06
3	168.9	0.14

Table 1

(a) By considering the frequency response function as a modal sum, show that the Q-factor may be estimated from the amplitude of the first mode and the pseudo-static response. Calculate this Q-factor and hence estimate the half-power bandwidth for each mode, assuming the damping in the beam is independent of frequency. [20%]

(b) Sketch the magnitude of the displacement frequency-response function using a decibel vertical axis. [25%]

(c) Sketch the corresponding Nyquist plot for the *velocity* frequency-response function, highlighting the important features. [25%]

(d) The shaker is moved from the tip of the cantilever to a position close to, but not at, the root. Indicate on your sketches for part (a) and part (b) how you expect the functions to change. [20%]

(e) It is suggested to use an impulse hammer, rather than a shaker, to excite the beam. Describe the potential advantages and disadvantages of doing this. [10%]

4 A ski manufacturer seeks to maximise the damping of a ski by investigating material and construction choices.

(a) Use Rayleigh's principle together with the 'correspondence principle' to derive an expression for the Q-factor of a uniform Euler beam, and estimate values for each of the following materials:

(i) carbon fibre (CFRP)

(ii) titanium

(iii) wood [20%]

(b) It is proposed that a ski could be constructed using a symmetric three-layer design, such that a core material is sandwiched by an identical pair of top and base layers. In this case, the 'three-layer' formula for the effective bending stiffness EI simplifies to the following:

$$EI = E_1 \left\{ \frac{h_1^3}{6} + \frac{h_1 (h_1 + h_2)^2 g}{1 + 2g} \right\} + E_2 \left\{ \frac{h_2^3}{12} - \frac{h_2^2 (h_1 + h_2)}{12(1 + 2g)} \right\}$$

For a wooden core with CFRP top and base layers, estimate the effective loss factor when the wavelength is approximately 2 m. Use material properties from the datasheet listing your selected values, together with the following: $h_1 = 0.5$ mm, $h_2 = 10$ mm, $G_2 = E_2/3$ and assume that the shear loss factor for wood is $\eta_G = 0.1$. [20%]

(c) Assume now that damping is dominated by shear in the core, i.e. assume $\eta_G = 0.1$ and that E_1 and E_2 are real-valued. What value for the real part of G_2 results in the maximum imaginary part of EI, and what is the corresponding effective Q-factor? You may use the result that:

$$\text{imag}\,(EI) \approx \frac{\partial EI}{\partial g} g \eta_G$$

[50%]

(d) Comment on the effectiveness of the three-layer design in terms of damping performance. [10%]

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Part IIB Data Sheet

Module 4C6 Advanced Linear Vibration

1 Vibration Modes and Response

Discrete Systems

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems

1. Equation of motion

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 2 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

See Section 2 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 2) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

1.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\widetilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 2.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\widetilde{T}}$ is *stationary* near each mode.

2 Governing equations for continuous systems

2.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

2.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$
\begin{aligned}
\cos\alpha\cosh\alpha + 1 = 0, &\quad \alpha_1 = 1.8751 \\
\cos\alpha\cosh\alpha - 1 = 0, &\quad \alpha_1 = 4.7300 \\
\tan\alpha - \tanh\alpha = 0, &\quad \alpha_1 = 3.9266
\end{aligned}
$$

Some devices for vibration excitation and measurement

Moving coil electro-magnetic shaker

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

Piezo stack actuator

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1μm, u
nloaded resonance 400kHz

Impulse hammer

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

Piezo accelerometer

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

MEMS accelerometer

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

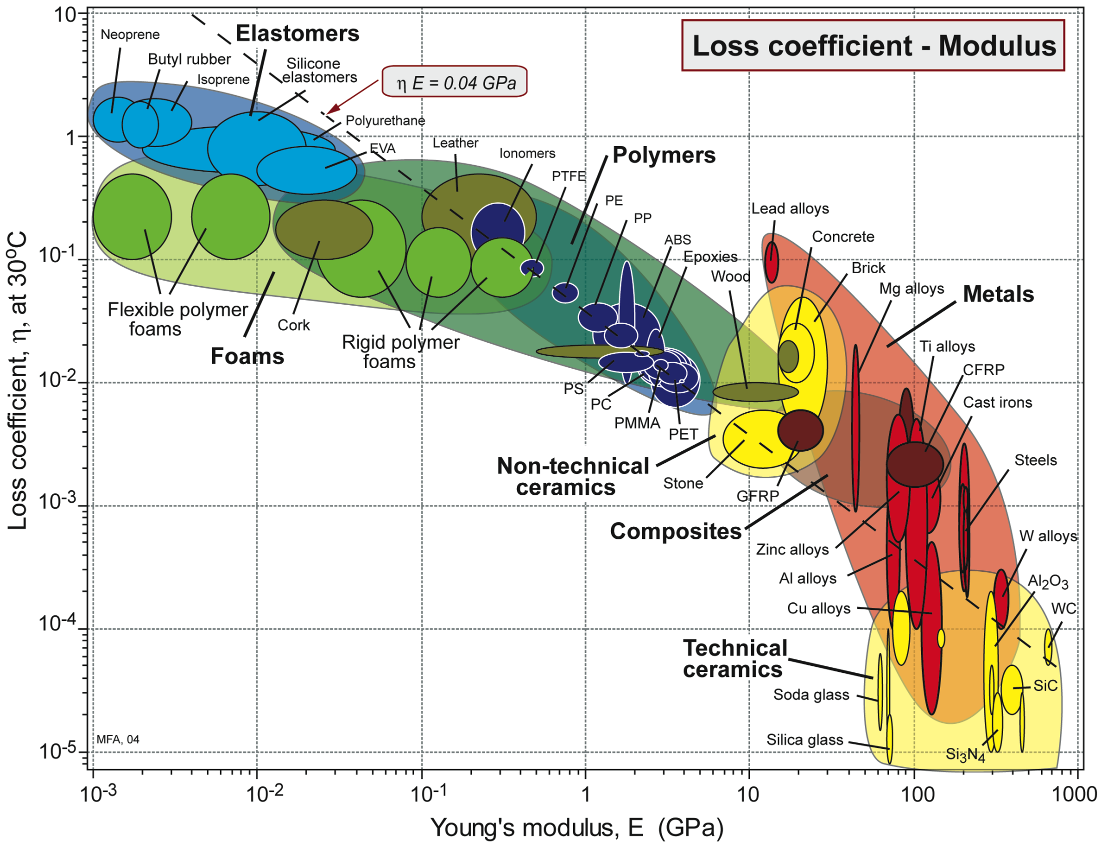
Laser Doppler Vibrometer

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \to E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



For a complex natural frequency ω:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh}\right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1}.$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12}\left[\frac{h_{31} - d}{1+g}\right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2}(h_{21} - d) + E_3 h_3 (h_{31} - d)\right]\left[\frac{h_{31} - d}{1+g}\right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t C \, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.85a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r, θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n=0$	$n=1$	$n=2$	$n=3$
$ka =$	2.404	3.832	5.135	6.379
$ka =$	5.520	7.016	8.417	9.760
$ka =$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}\,.$$