Version JPT/4

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 23 April 2024   2 to 3.40

Module 4C6

ADVANCED LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (9 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The lateral response of a three-storey building may be described using the model shown in Fig.1. All three floors have the same mass m, and the lateral stiffness between floors is $2k$. As part of a full-scale dynamic test, accelerometer measurements are made on each floor whilst the building responds to ambient excitation from the wind. Initial Fourier analysis indicates the natural frequencies to be 3.8 Hz, 10.7 Hz and 15.4 Hz.

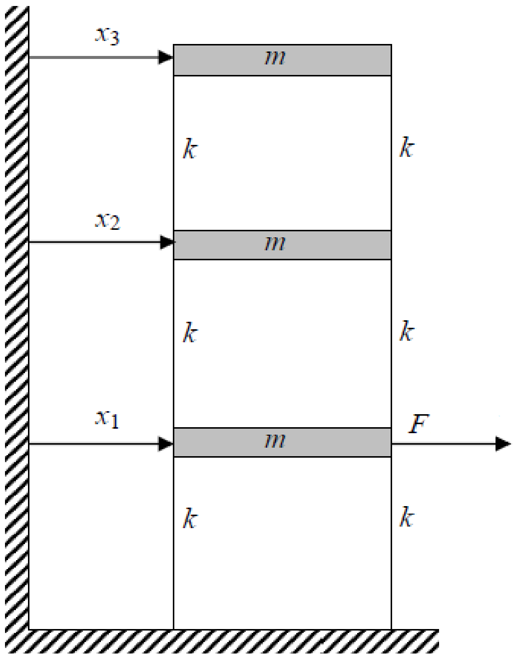


Fig. 1

(a) State the fundamental assumptions necessary to have confidence in the identified natural frequencies and explain why such measurements are unsuitable for measuring the mode shapes of the building using standard modal analysis techniques. [20%]

(b) It is suggested that an impulse hammer is used to measure the frequency-response functions for excitation applied to the bottom floor. Sketch the anticipated mode shapes and hence, using a decibel vertical axis, sketch the anticipated magnitude of the acceleration frequency-response function for each floor over a frequency range that includes all three natural frequencies. You are not expected to calculate the modes. [35%]

(c) Sketch the corresponding Nyquist plot of the frequency-response function for the top floor with excitation applied to the bottom floor. Identify each of the modal circles. [15%]

(d) The proposed impulse hammer has a tip mass of 5 kg.

(i) Calculate the minimum tip stiffness for the hammer to be potentially useful for identifying the first three mode shapes. [15%]

(ii) Explain why an impulse hammer may not be a suitable choice of excitation in this case and discuss possible alternatives. [15%]

2 (a) Figure 2 shows three masses m_i that are free to move on a frictionless circular track in the horizontal plane. At equilibrium, the masses are equally spaced around the track. The circumferential displacements of the masses are denoted x_i. Adjacent masses m_i and m_j are connected to each other via springs of stiffness k_{ij}, defined such that the tangential force that each spring applies to its connected masses is $f = \pm k_{ij}(x_i - x_j)$. Initially all masses and stiffnesses are equal to m and k respectively.

(i) Find the mass and stiffness matrices **M** and **K** of the system. [10%]

(ii) Find the mode shapes and natural frequencies. [10%]

(b) Damping is now included in the model by a set of viscous dashpots connecting the masses in parallel with the springs. The damping rate between masses 2 and 3 is twice that of the others, i.e. $c_{12} = c_{31} = c$ and $c_{23} = 2c$. Subscripts follow the notation for the springs k_{ij} and correspond to the adjacent masses.

(i) Find an expression for the Rayleigh dissipation function and hence find the damping matrix **C**. [10%]

(ii) Write the equation of motion for the damped system in first order form, $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$. Define **z** and write **A** in terms of **M**, **C** and **K**. [5%]

(iii) Without calculation, describe what you expect for the eigenvalues of **A**. [15%]

(c) The dashpots from part (b) are removed. The spring k_{23} is replaced with a complex stiffness that has damping modelled using a loss factor γ, such that $k_{23} = k(1 + i\gamma)$. The other springs are undamped. Using the correspondence principle together with Rayleigh's principle, estimate the Q-factor for each mode. [50%]

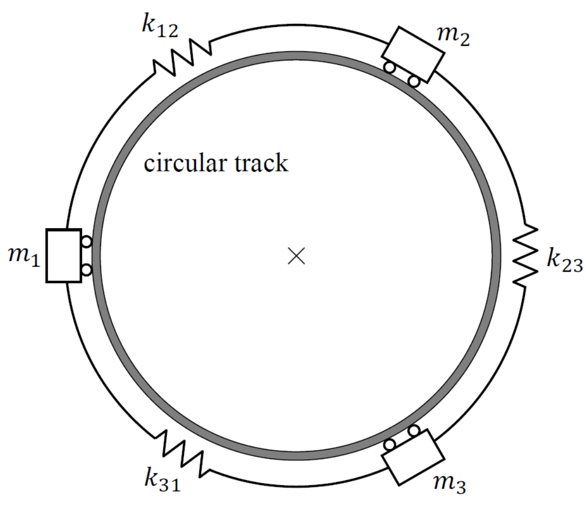


Fig. 2

3 (a) Figure 3(a) shows a thin-walled enclosure of volume V. A hole of effective radius a_1 and area A_1 is made at one end. The 'air-plug' within the hole is indicated by the dashed line (not to scale), and its displacement is denoted x_1, noting the sign convention shown in the figure.

(i) By considering the force f_1 that acts on the air-plug given a displacement x_1, derive an expression for the effective stiffness k_1 of the air volume. Recall that $p = -Kv/V$, where p is the pressure change inside the volume, K is the bulk modulus of air and v is the small change in volume V that arises from the displacement x_1. [20%]

(ii) Using your result from (a)(i) or otherwise, derive the natural frequency for the Helmholtz oscillator. The speed of sound in air is given by $c^2 = K/\rho$, where ρ is the density of air. [20%]

(b) A second circular hole of effective radius a_2 and area A_2 is now made at the opposite end of the enclosure, as shown in Fig. 3(b). The displacement of the second air-plug is denoted x_2, noting the sign convention shown in the figure.

(i) From Newton's second law and by considering the forces f_1 and f_2 that act on the air-plugs given displacements x_1 and x_2, or otherwise, derive the equation of motion for the two-hole system. Write your answer in the form:

$$\mathbf{M}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \mathbf{K}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix},$$

where **M** is the mass matrix for the air-plugs and **K** is the effective stiffness matrix of the air volume. [30%]

(ii) For the symmetric case when both holes have effective radius a and area A, find the mode shapes and natural frequencies. [20%]

(iii) Comment on how your answers to parts (a)(ii) and (b)(ii) are consistent with the interlacing theorem. [10%]

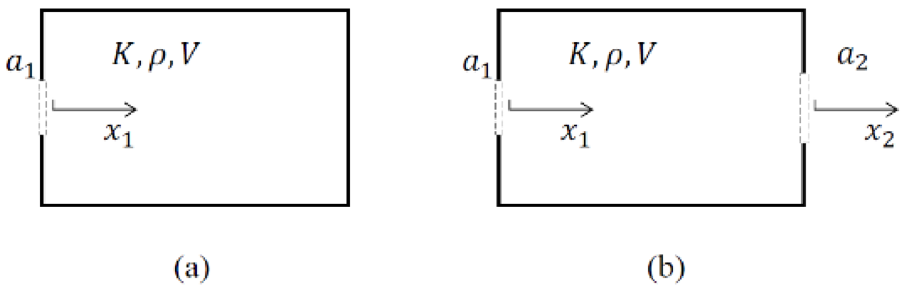


Fig. 3

4 (a) A free-free beam is made from a material of Young's modulus E and density ρ. The cross-sectional area is A and the second moment of area is I.

(i) Describe briefly the main sources of damping for beams made of steel, ceramic, and a thermosetting plastic. [20%]

(ii) Suggest two experimental methods for estimating the material damping of the beam, and identity potential problems that may arise when trying to make these measurements. [20%]

(iii) Making use of the Ashby materials chart in the datasheet (showing loss coefficient and Young's modulus), estimate the likely range of Q-factors for the modes of the beam if it was made from steel. [10%]

(b) The beam rests on a uniform elastic foundation that has stiffness K per unit length, as illustrated in Fig. 4. The expression for the potential energy V of the beam on an elastic foundation is given by:

$$V = \frac{1}{2}EI\int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}K\int_0^L y^2 \mathrm{d}x,$$

where y is the transverse displacement of the beam.
The material loss factors of the beam and elastic foundation are η_E and η_K respectively.

(i) Derive an expression for the modal Q-factors of the beam. Write your answer in the form:

$$Q_n = (J_1\eta_E + J_2\eta_K)^{-1}$$

and find expressions for J_1 and J_2. [40%]

(ii) Comment on the factors that affect the relative contribution of damping from the beam and foundation materials. [10%]

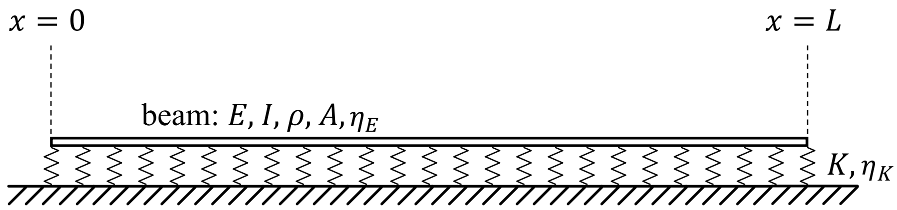


Fig. 4

END OF PAPER

Part IIB Data Sheet

Module 4C6 Advanced Linear Vibration

1 Vibration Modes and Response

Discrete Systems

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems

1. Equation of motion

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 2 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

See Section 2 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 2) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos \omega_n t \Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t \Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos \omega_n t \Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t \Big]$$

for $t \geq 0$ (with small damping).

1.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\widetilde{T}} = \frac{\mathbf{y}^T\mathbf{K}\mathbf{y}}{\mathbf{y}^T\mathbf{M}\mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 2.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\widetilde{T}}$ is *stationary* near each mode.

2 Governing equations for continuous systems

2.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

2.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

2.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$
\begin{aligned}
\cos\alpha\cosh\alpha + 1 = 0, &\quad \alpha_1 = 1.8751 \\
\cos\alpha\cosh\alpha - 1 = 0, &\quad \alpha_1 = 4.7300 \\
\tan\alpha - \tanh\alpha = 0, &\quad \alpha_1 = 3.9266
\end{aligned}
$$

Some devices for vibration excitation and measurement

Moving coil electro-magnetic shaker

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

Piezo stack actuator

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1μm, u
nloaded resonance 400kHz

Impulse hammer

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

Piezo accelerometer

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

MEMS accelerometer

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

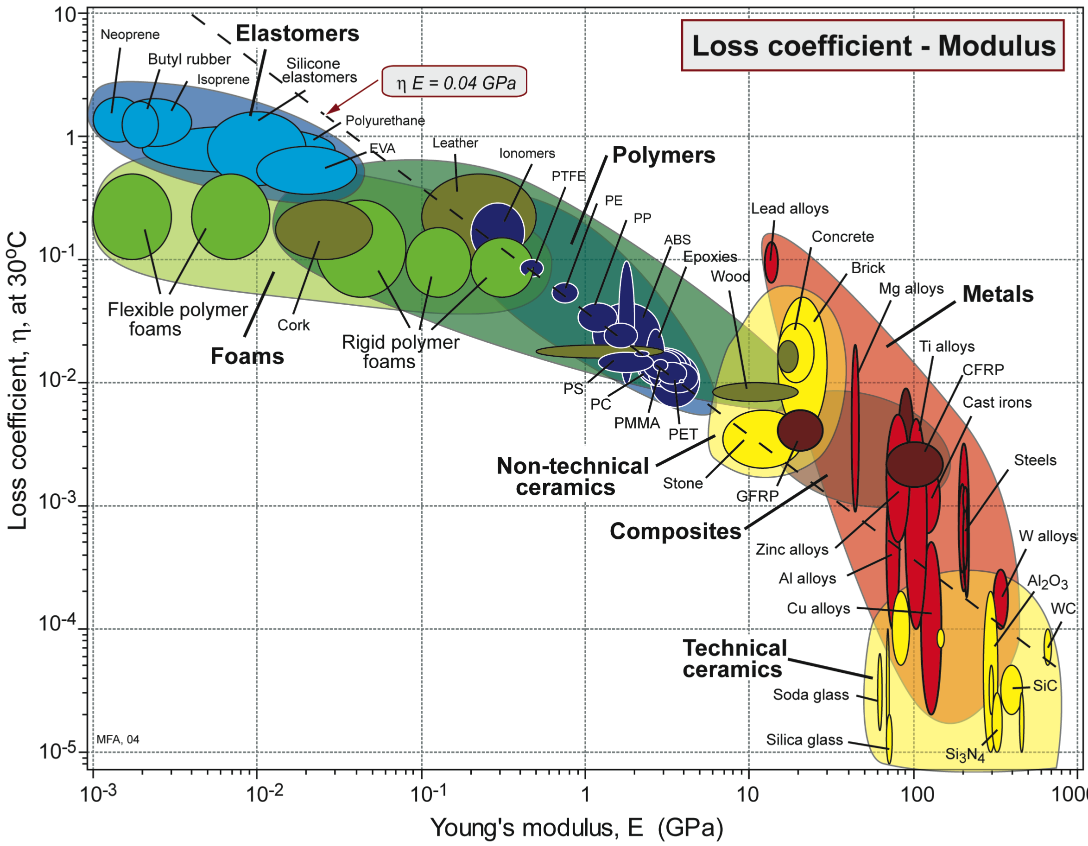
Laser Doppler Vibrometer

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \to E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



For a complex natural frequency ω:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh} \right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1} .$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12} \left[\frac{h_{31} - d}{1+g} \right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2} (h_{21} - d) + E_3 h_3 (h_{31} - d) \right] \left[\frac{h_{31} - d}{1+g} \right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t \, C \, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.85a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r, θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta \, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$ka =$	2.404	3.832	5.135	6.379
$ka =$	5.520	7.016	8.417	9.760
$ka =$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}\ .$$