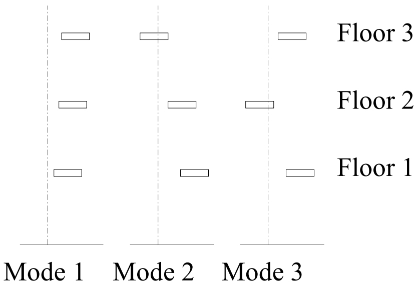
Q1

a) To have confidence in the identified natural frequencies, the wind must be sufficiently broadband to excite the three modes to a measurable response level. There must also be no additional excitation with frequency content that may be confused with the natural frequencies (e.g. machinery running in the building or vehicles on a nearby road). [2]

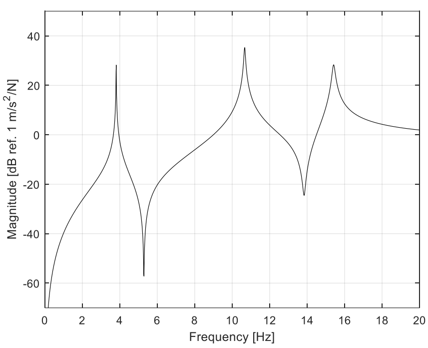
Such measurements are unsuitable for measuring the mode shapes because no measurements of the excitation force are made. Only the operating shapes of the building may be identified. [2]

b) This is the familiar building model from Part I. The anticipated mode shapes are:



[2]

The phases of the floor responses [Floor 1 Floor 2 Floor 3] are: [+ + +], Mode 1; [+ + -], Mode 2; and [+ - +], Mode 3. Hence, for each floor in turn, accounting for the sign changes between the modal contributions, the FRFs are as follows.

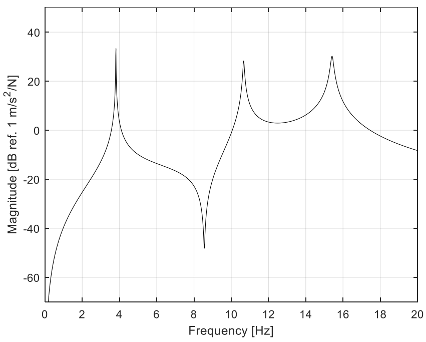


Floor 1

Driving point response, H_{11}

No sign changes .: anti-resonances between all peaks

[3]

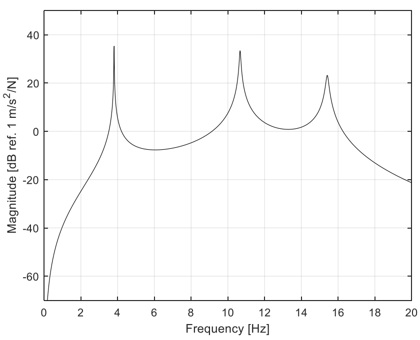


Floor 2

Transfer function, H_{21}

Sign change between second and third modes .: one anti-resonance between the second and third peaks

[1]



Floor 3

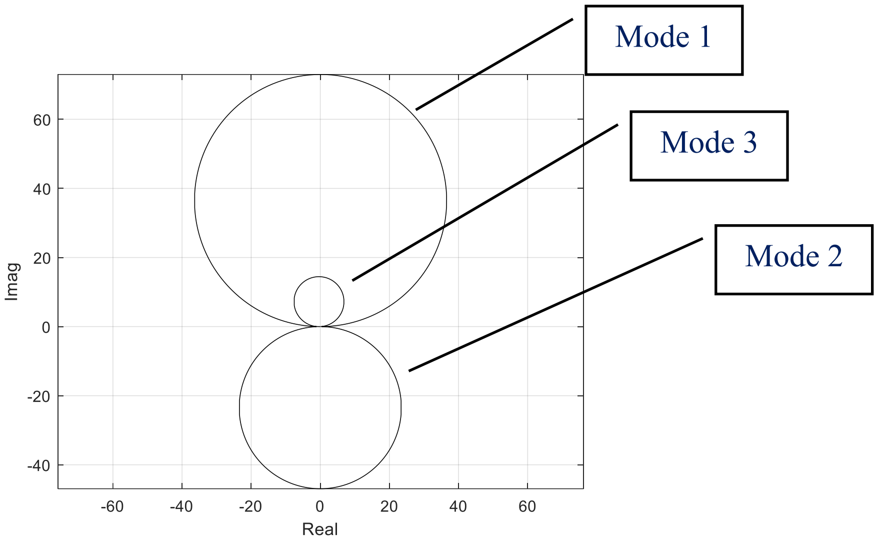
Transfer function, H_{31}

Signs alternate between modes .: no anti-resonances

[1]

Full marks are obtained by correctly labelling the axes, sketching the low-frequency form and accounting for anti-resonances between resonance peaks. The relative amplitudes are not expected.

b)



The response is measured on the top floor. Modes 1 and 3 therefore involve responses that are in-phase with the excitation, hence circles on the positive imaginary axis. Mode 2 involves an out-of-phase response, hence a circle on the negative imaginary axis.

Full marks are obtained by correctly labelling and positioning the circles on the imaginary axis. The relative amplitudes are not expected. [3]

c) (i) The hammer must impart sufficient frequency content to excite all three modes. Mode 3 lies at 15.4 Hz so aim to excite up to, say, 20 Hz.

Hammer concentrates energy up to $\approx 2\Omega$, $\Omega = \sqrt{K/M}$, with Ω = 10 Hz

$\therefore\ K = \Omega^2 M = (2\pi.10)^2.5 \approx 20$ kN/m

[3]

(ii) Even if the required frequency content is achieved, the level of excitation from such a small hammer is unlikely to be sufficient given the mass and damping of a real building.

The only alternative is a large shaker, which might just provide sufficient excitation force, but even this is unlikely. The only option may be the operational modal analysis conducted initially! [3]

4C6 draft sols

Q2 (a)(i) $M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

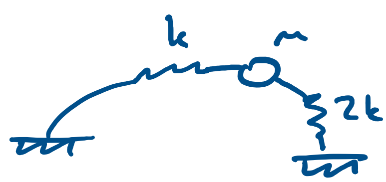
$$V = \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}k(x_3 - x_2)^2 + \tfrac{1}{2}k(x_1 - x_3)^2$$

$$\Rightarrow K = k\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(ii) Rigid-body mode: $u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\omega_0 = 0$

Symmetry $\Rightarrow$ $u_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\omega_1 = \sqrt{3k/m}$

as equiv to:



& circular symmetry means orientation arbitrary, so:

$u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\omega_2 = \sqrt{3k/m}$

(or $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, or any linear combination)

(b) (i) $R = \frac{1}{2}c(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2}c(\dot{x}_1 - \dot{x}_3)^2 + \frac{1}{2}2c(\dot{x}_2 - \dot{x}_3)^2$

$$\Rightarrow C = c\begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$

(ii) $\dot{z} = Az$ where $z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$

& $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$ from datasheet.

(iii) undamped $\omega_n^2 = 0,\ 3k/m,\ 3k/m$

damped: $\zeta_n = 0,\ \zeta_1,\ \zeta_2$

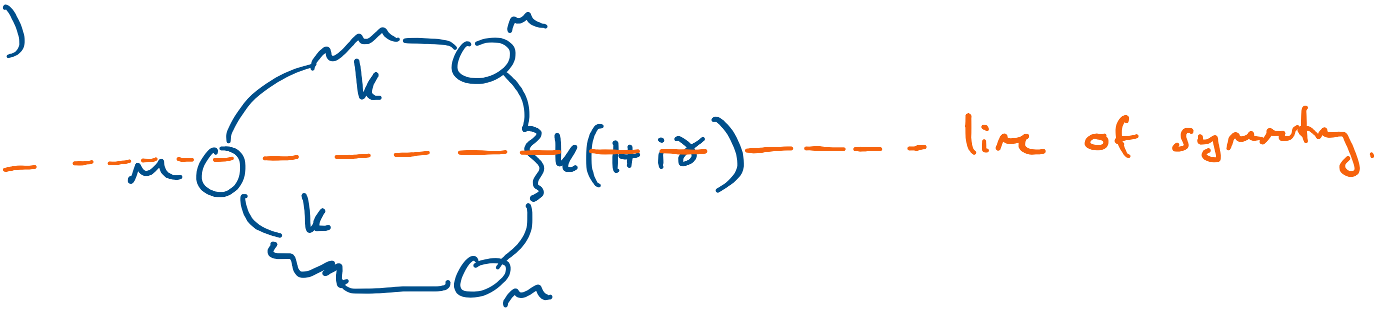
Expect 6 evals, real part $= \zeta_n \omega_n'$, imag $= \pm\omega_n'$, conj pairs.

$\lambda_1 = 0, \quad \lambda_2 = 0,$

$\lambda_3 = \sqrt{3k/m}\,(\zeta_1 + i), \quad \lambda_4 = \sqrt{3k/m}\,(\zeta_1 - i)$

$\lambda_5 = \sqrt{3k/m}\,(\zeta_2 + i), \quad \lambda_6 = \sqrt{3k/m}\,(\zeta_2 - i)$

(c)



Circular symmetry broken, but line of symmetry remains.

Mode shapes now aligned to this line of symmetry.

Mode shapes symmetric or antisymmetric.

Rigid body mode unchanged.

Modes : $u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\omega_0 = 0$, $Q = \infty$ // unchanged.

need $$K = k \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1+(1+i\gamma) & -(1+i\gamma) \\ -1 & -(1+i\gamma) & 1+(1+i\gamma) \end{bmatrix}$$

$$= k \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2+i\gamma & -(1+i\gamma) \\ -1 & -(1+i\gamma) & 2+i\gamma \end{bmatrix} \quad //$$

For mode ① : $u_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ from symmetry.

$$R = \frac{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2+i\gamma & -(1+i\gamma) \\ -1 & -(1+i\gamma) & 2+i\gamma \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} k}{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} m}$$

$$= \frac{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2+i\gamma + 1+i\gamma \\ -(1+i\gamma) - (2+i\gamma) \end{bmatrix} k}{2m}$$

$$= \frac{k\left(3 + 2i\gamma + 3 + 2i\gamma\right)}{2m} = \frac{(6 + 4i\gamma)k}{2m} = \frac{k}{m}(3+2i\gamma)$$

hence $\omega_1'^2 = \frac{k}{m}(3+2i\gamma) = \frac{3k}{m}\left(1 + \frac{2i\gamma}{3}\right)$

$$\omega_1' \simeq \sqrt{\frac{3k}{m}}\left(1 + \frac{i\gamma}{3}\right)$$

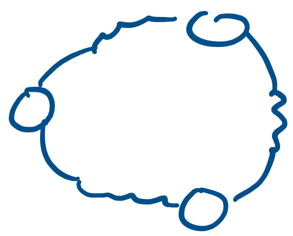
from free decay: $e^{i\omega_1' t} = e^{i\omega_1 t} \cdot e^{-\omega_1 \gamma/3\, t}$

compare: $e^{-\zeta_n \omega_n t}$

$\Rightarrow \zeta_1 = \gamma/3, \quad Q_1 = \frac{3}{2\gamma}$ //

(2) Symmetry broken, so orthogonality holds:

find $u_2 \perp \begin{bmatrix}1\\1\\1\end{bmatrix}$ & $\begin{bmatrix}0\\1\\-1\end{bmatrix}$



let $u_2 = \begin{bmatrix}1\\a\\b\end{bmatrix}$: $\begin{bmatrix}1 & a & b\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 0.$

$\Rightarrow \quad 1 + a + b = 0.$

$\begin{bmatrix}1 & a & b\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0.$

$\Rightarrow \quad a - b = 0$

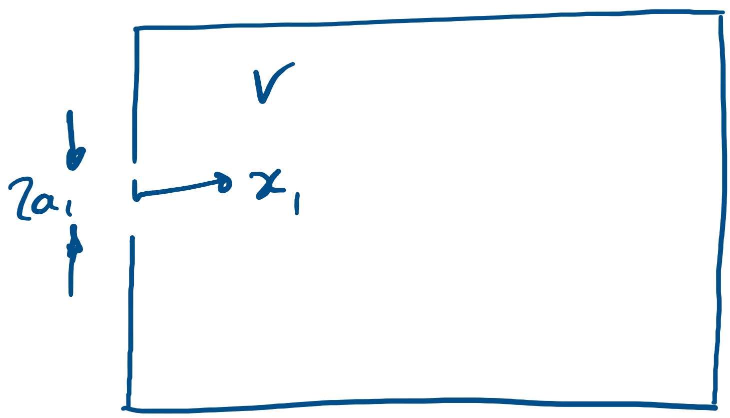
$\Rightarrow \quad u_2 = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ (scale factor not important).

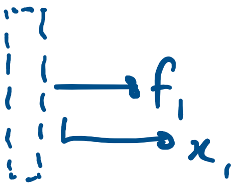
No relative motion between 2 & 3 so damping = 0.

$$S_2 = 0 \quad \& \quad Q_2 = \infty$$

$$Q_0 = \infty, \quad Q_1 = \frac{3}{28}, \quad Q_2 = \infty$$

Q3/ (a)





$$f_1 = -k_1 x_1$$

[define f_1 as force on air-plug due to x_1]

$$f_1 = -p A_1$$

$$p = -K \frac{dV}{V}$$

$$dV = -A_1 x_1$$

so $$f_1 = -\underbrace{\frac{K}{V} A_1^2}_{k_1} x_1$$

so $$k_1 = \frac{K A_1^2}{V}$$ //

(ii) $$\omega_n^2 = k_1/m = \frac{(K . A_1^2/V)}{\rho A_1 . L_1}$$

$$= \frac{K A_1}{\rho V L_1} = \frac{c^2 A_1}{V L_1}$$

thin-wall ⇒ double flange, $L = 1.7 a_1$, $\omega_n^2 = \dfrac{c^2 A_1}{1.7 a_1 V}$

(b)

$$m_1 = \rho A_1 L_1 \qquad m_2 = \rho A_2 L_2$$

$f_1 = -p A_1 \longrightarrow f_1$ & x_1 defined in same sense

$$p = -K \frac{dV}{V}$$

$$dV = -A_1 x_1 + A_2 x_2$$

so $$f_1 = \frac{KA_1}{V}\left(-A_1 x_1 + A_2 x_2\right) = -\frac{KA_1^2}{V} x_1 + \frac{KA_1A_2}{V} x_2$$

now $f_2 = p A_2 \longrightarrow f_2$ & x_2 defined in same sense.

$$p = -K \frac{dV}{V}.$$

dV as above.

so $$f_2 = -\frac{KA_2}{V}\left(-A_1 x_1 + A_2 x_2\right) = \frac{KA_1A_2}{V} x_1 - \frac{KA_2^2}{V} x_2$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{K}{V} \begin{bmatrix} -A_1^2 & A_1 A_2 \\ A_1 A_2 & -A_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Apply F=ma:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \rho A_1 L_1 & 0 \\ 0 & \rho A_2 L_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{K}{V} \begin{bmatrix} A_1^2 & -A_1 A_2 \\ -A_1 A_2 & A_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $L_1 = 1{\cdot}7a_1$, $L_2 = 1{\cdot}7a_2$

(ii) Symmetric case, modes symm or antisymm:

Rigid body mode: $\omega_0 = 0$, $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Antisymmetric mode: $\omega_1^2 = 2\dfrac{c^2 A}{VL}$, $u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

as node in centre
so volume is effectively half.

(iii) 2-holes: $\omega = 0, 2\omega_1$ / 1-hole: $\omega = \omega_1$

added constraint (blocking hole)
$\Rightarrow$ new frequency is between original.

Q3/ (a)(i) Material damping:

Steel: dislocation movement

Ceramic: usually very low due to strong ionic & covalent bonds, unless cracked.

Thermoplastic: weak bonds between polymer chains continually break & reform under cyclic loading, causing energy loss.

For a beam, additional damping from:

- Fluid structure interaction
- Boundary damping: friction
  air pumping
  microslip

(ii)

- Tensile testing machine under cyclic loading
  - measure force-displacement (stress-strain) for a material specimen.
  - find gain & phase => $E(1+i\eta)$
  - repeat for frequencies of interest.

Problems: hard to avoid boundary damping.
bandwidth often limited.

- Find modal damping factors from beam measurements & infer η from these assuming sinusoidal mode shapes.

Problems: only obtain data @ ω_n
requires mode shape assumption.

(iii) $$\omega_n^2 = \frac{V}{\tilde{T}} = \frac{\frac{1}{2} EI \int_0^L u''^2 \, dx}{\frac{1}{2} \rho A \int_0^L u^2 \, dx}$$

$$E \to E(1 + i\eta_E), \quad \omega_n'^2 = \omega_n^2 (1 + i\eta_E)$$

$$\omega_n' \simeq \omega_n \left(1 + \tfrac{1}{2} i\eta_E\right)$$

free response: $y \propto e^{i\omega_n t} \cdot \underbrace{e^{-\frac{1}{2}\omega_n \eta_E t}}_{\text{cf } e^{-\zeta\omega_n t}}$

$$\Rightarrow \zeta = \eta_E / 2.$$

$$\& \quad Q = \frac{1}{\eta_E}$$

for steel: $\eta_E = 2 \times 10^{-4} - 3 \times 10^{-3}$ from datasheet

so approx: $Q = 300 - 5000$

(b)(i)

$$\omega_n^2 = \frac{\overbrace{\tfrac{1}{2}EI\int u''^2\,dx}^{V_E} + \overbrace{\tfrac{1}{2}K\int u^2\,dx}^{V_K}}{\underbrace{\tfrac{1}{2}\rho A\int u^2\,dx}_{\tilde{T}}}$$

$$E \to E(1+i\eta_E), \quad K \to K(1+i\eta_K)$$

$$\omega_n'^2 = \frac{V_E(1+i\eta_E) + V_K(1+i\eta_K)}{\tilde{T}}$$

$$= \underbrace{\frac{V_E+V_K}{\tilde{T}}}_{=\,\omega_n^2} \cdot \underbrace{\frac{\tilde{T}}{V_E+V_K}}_{=\,1/\omega_n^2} \cdot \left[\frac{V_E}{\tilde{T}}(1+i\eta_E) + \frac{V_K}{\tilde{T}}(1+i\eta_K)\right]$$

$$= \omega_n^2\left[J_1(1+i\eta_E) + J_2(1+i\eta_K)\right]$$

where: $J_1 = \dfrac{V_E}{\omega_n^2\tilde{T}}$, $J_2 = \dfrac{V_K}{\omega_n^2\tilde{T}}$

$$\omega_n'^2 = \omega_n^2\left(\underbrace{J_1 + J_2}_{1} + i(\eta_E J_1 + \eta_K J_2)\right)$$

$$\omega_n' \simeq \omega_n\left(1 + \tfrac{1}{2}i(\eta_E J_1 + \eta_K J_2)\right)$$

$$\Rightarrow \zeta_n = \tfrac{1}{2}i(\eta_E J_1 + \eta_K J_2)$$

so $Q_n = \dfrac{1}{\eta_E J_1 + \eta_K J_2}$ & J_1, J_2 defined above

(ii) $J_1/J_2 = V_E/V_K = \dfrac{\frac{1}{2}EI\int u''^2\,dx}{\frac{1}{2}K\int u^2\,dx}$

$$= \frac{EI}{K}\cdot\frac{\int\left(\frac{n\pi}{L}\right)^2\sin^2\frac{n\pi x}{L}\,dx}{\int\sin^2\frac{n\pi x}{L}\,dx}$$

$$= \left(\frac{n\pi}{L}\right)^2\frac{EI}{K}$$

So beam damping dominates at high frequency & when bending stiffness is large compared with foundation stiffness