EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 29 April 2025 9.30 to 11.10

Module 4C6

ADVANCED LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (9 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Figure 1(a) shows an illustration of a simplified Ocarina: a musical instrument that uses the principle of a Helmholtz oscillator in order to produce a note. A mouthpiece (not shown) produces an unsteady airflow which excites the resonant modes of the acoustic cavity. The cavity is thin-walled, its volume is V, and there is a collection of circular holes. The sound hole has radius a_1 and is always open. There are also $N-1$ additional holes that can be closed by covering them up with a finger, which changes the pitch of the note produced. The radius of the i th hole is a_i and the corresponding outward displacement of the 'air-plug' is x_i.

(i) Derive an expression for the natural frequency of the instrument when all holes are closed except for the sound hole. The speed of sound in air is given by $c^2 = K/\rho$, where ρ is the density of air. Recall that $p = -Kv/V$, where p is the pressure change inside the volume, K is the bulk modulus of air and v is the small change in volume V that arises from the displacement x_1. [20%]

(ii) Now consider when all N holes are open. By applying Newton's second law to the air in the i th hole, or otherwise, derive the equation of motion for the N-hole instrument. Write the equation in the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{k}^{\mathrm{T}}\mathbf{x} = 0$$

where $\mathbf{x}$ is a vector of outward displacements of each 'air-plug' within each hole. Find the $N \times N$ mass matrix $\mathbf{M}$ and the $N \times 1$ vector $\mathbf{k}$ that can be used to construct the stiffness matrix $\mathbf{K} = \mathbf{k}\mathbf{k}^{\mathrm{T}}$. [30%]

(iii) Show that $\mathbf{u} = \mathbf{M}^{-1}\mathbf{k}$ is a mode shape, and find the corresponding natural frequency. By considering the form of the mass and stiffness matrices, find the other natural frequencies. [30%]

(iv) What happens to the pitch of a note when a hole is closed? How is this consistent with the interlacing theorem? [10%]

(b) An alternative design is proposed, consisting of a series of acoustic volumes separated by circular holes, as illustrated in Fig. 1(b). If the total number of holes is N, how many non-zero natural frequencies do you expect? Justify your answer. [10%]

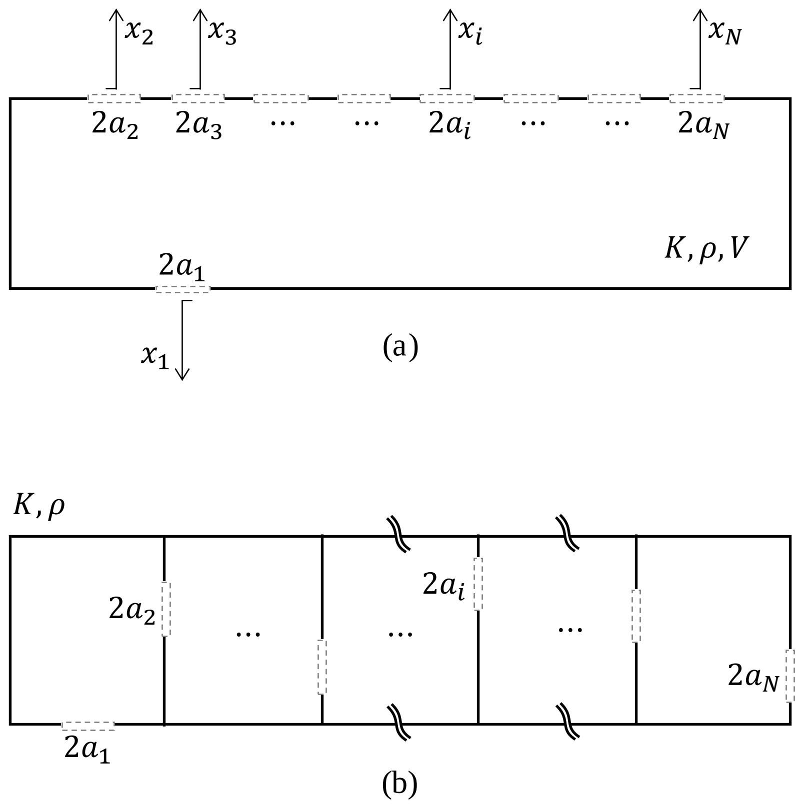

Fig. 1

2 A cylindrical duct of radius a and length L is open at both ends and has rigid walls. For sound inside the tube, acoustical pressure p is governed by a differential equation which can be written in cylindrical polar coordinates (r, θ, z) in the form:

$$c^2 \left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} \right] = \frac{\partial^2 p}{\partial t^2}$$

where c is the speed of sound. The boundary condition at the walls is:

$$\frac{\partial p}{\partial r} = 0$$

(a) For a sinusoidal wave propagating along the length of the duct, the pressure can be written in the form:

$$p = f(r, \theta)\mathrm{e}^{i(\omega t - \lambda z)}$$

Show that cross-sectional component of the mode shape $f(r, \theta)$ is governed by a similar equation to that for transverse vibration of a circular membrane. [10%]

(b) Use separation of variables to show that the mode shapes of the duct can be written in the form:

$$p = J_n(kr)g(n\theta)\,(A \sin \lambda z + B \cos \lambda z)$$

where J_n is a Bessel function. Find the function $g(n\theta)$ and the constants A, B, λ and the relationship between k, λ and ω. [30%]

(c) Sketch the cross-sectional component $f(r, \theta)$ of the mode shapes and label them with the corresponding values of ka and n. Find all cross-sectional mode shapes up to and including one nodal circle and two nodal diameters. A plot of the first four Bessel functions is shown in Fig. 2, with zero-crossings and stationary points labelled. [30%]

(d) Without detailed calculation, and making suitable assumptions, describe and sketch the mode shapes you would expect for the following systems.

(i) The acoustic pressure within a cuboid-shaped (rectangular) room. [10%]

(ii) A closed circular hoop made from a curved uniform beam, considering its radial displacement in the plane of the hoop. [10%]

(iii) A simply-supported circular plate, i.e. zero displacement and zero moment at the boundary. [10%]

 (cont.

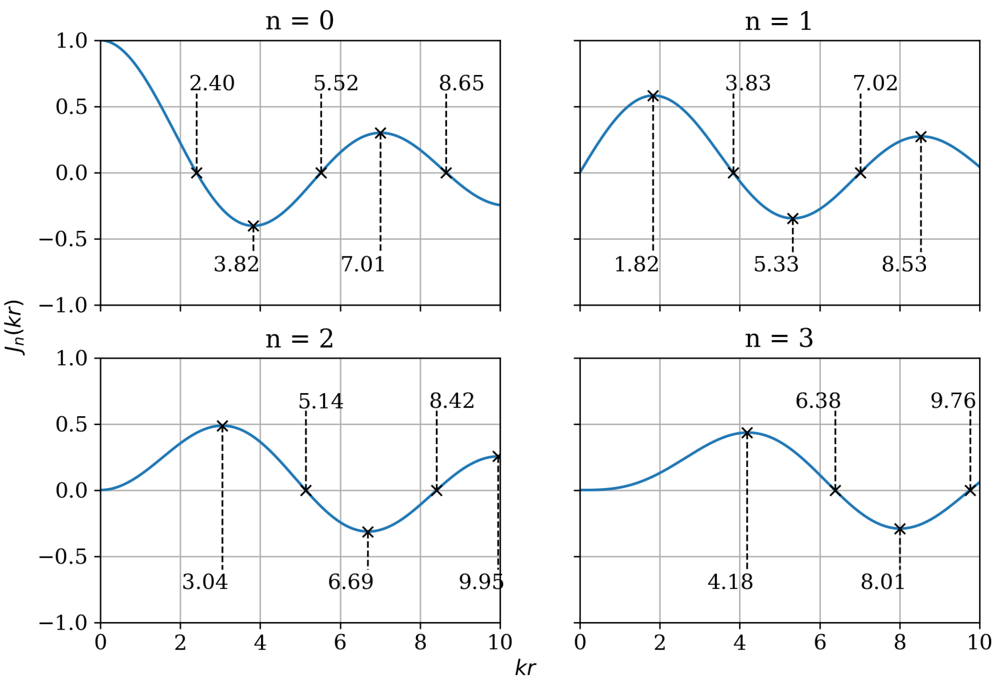

Fig. 2

(TURN OVER

3 Figure 3 shows a vibrating system with two degrees of freedom, with masses m, grounded springs of stiffness k and a coupling spring of stiffness s.

(a) Find the mass and stiffness matrices $\mathbf{K}, \mathbf{M}$, mode shapes $\mathbf{u}_n$, and expressions for the natural frequencies ω_n of the undamped system. [10%]

(b) The mode shapes of the undamped system satisfy $\left(\mathbf{K} - \omega_n^2\mathbf{M}\right)\mathbf{u}_n = 0$. Show that if Rayleigh damping is assumed, such that $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$ where α and β are constants, then the mode shapes are unchanged by damping. [20%]

(c) Rayleigh damping is used to model energy loss in the system, with $\alpha = \beta = 0.1$, $m = 2$, $k = 3$ and $s = 1$.

(i) Write down the equation of motion for unforced vibration of this damped system in first-order form $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$. Define $\mathbf{z}$ clearly and write out the full matrix $\mathbf{A}$. [20%]

(ii) Find the modal Q-factors for this system. [30%]

(d) If $\mathbf{C} = \alpha\mathbf{K}^2$ and $\mathbf{M} = \gamma\mathbf{I}$, where α and γ are constants, show whether or not the mode shapes will be affected by damping. [20%]

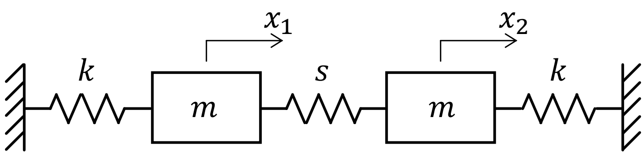

Fig. 3

4 (a) Experimental modal analysis is to be used to measure the transfer function between two locations on an aircraft wing panel. An impulse hammer is available with a mass of 0.2 kg and a choice of three hammer tips, of stiffness 0.5 MNm^{-1}, 2 MNm^{-1} and 10 MNm^{-1}. The frequency range of interest extends from 10 Hz to 1 kHz.

(i) Which of the three hammer tips is most suitable for this application and why? [15%]

(ii) The wing panel is at risk of damage if the hammer force exceeds 1.3 kN. Assuming a half-sinusoidal force pulse, estimate the maximum acceptable hammer impact velocity for your chosen hammer tip. [15%]

(iii) A low-pass filter is available within the data logger with a cut-off frequency of 1 kHz. State one advantage and one disadvantage of using such a filter. [10%]

(iv) Assuming the filter is selected, select a suitable sampling frequency for the data logger, justifying your choice. [10%]

(b) The modal analysis identifies the following parameters of the displacement frequency-response function (receptance) measured between the two locations.

Mode number n	Frequency (Hz)	Q-factor	$u_j^{(n)} u_k^{(n)}$ (kg^{-1})
1	200	50	2
2	650	100	-5

Labelling axes and marking important features, draw clear sketches of:

(i) the magnitude of the transfer function on a decibel scale; [25%]

(ii) the corresponding Nyquist plot for displacement; [15%]

(iii) the corresponding Nyquist plot for velocity. [10%]

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Part IIB Data Sheet

Module 4C6 Advanced Linear Vibration

1 Vibration Modes and Response

Discrete Systems / **Continuous Systems**

1. Equation of motion

Discrete Systems: The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

Continuous Systems: The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 2 for examples.

2. Kinetic Energy

Discrete Systems:

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

Continuous Systems:

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

Discrete Systems:

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

Continuous Systems: See Section 2 for examples.

4. Natural frequencies and mode shapes

Discrete Systems: The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

Continuous Systems: The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 2) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

Discrete Systems:

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems:

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_j q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos \omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \left[1 - \cos \omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_{n} \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \left[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\right]$$

for $t \geq 0$ (with small damping).

1.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{\tilde{T}}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 2.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{\tilde{T}}}$ is *stationary* near each mode.

2 Governing equations for continuous systems

2.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\frac{\partial^2 y}{\partial t^2} - P\frac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}P\int\left(\frac{\partial y}{\partial x}\right)^2 dx$	$T = \frac{1}{2}m\int\left(\frac{\partial y}{\partial t}\right)^2 dx$

2.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \frac{1}{2}GJ\int\left(\frac{\partial \theta}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho J\int\left(\frac{\partial \theta}{\partial t}\right)^2 dx$

2.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 y}{\partial t^2} - EA\frac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}EA\int\left(\frac{\partial y}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial y}{\partial t}\right)^2 dx$

2.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \frac{1}{2}EI\int\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
\cos\alpha\cosh\alpha + 1 = 0, &\quad \alpha_1 = 1.8751\\
\cos\alpha\cosh\alpha - 1 = 0, &\quad \alpha_1 = 4.7300\\
\tan\alpha - \tanh\alpha = 0, &\quad \alpha_1 = 3.9266
\end{aligned}$$

Some devices for vibration excitation and measurement

Moving coil electro-magnetic shaker

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

Piezo stack actuator

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1μm, u
nloaded resonance 400kHz

Impulse hammer

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

Piezo accelerometer

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

MEMS accelerometer

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

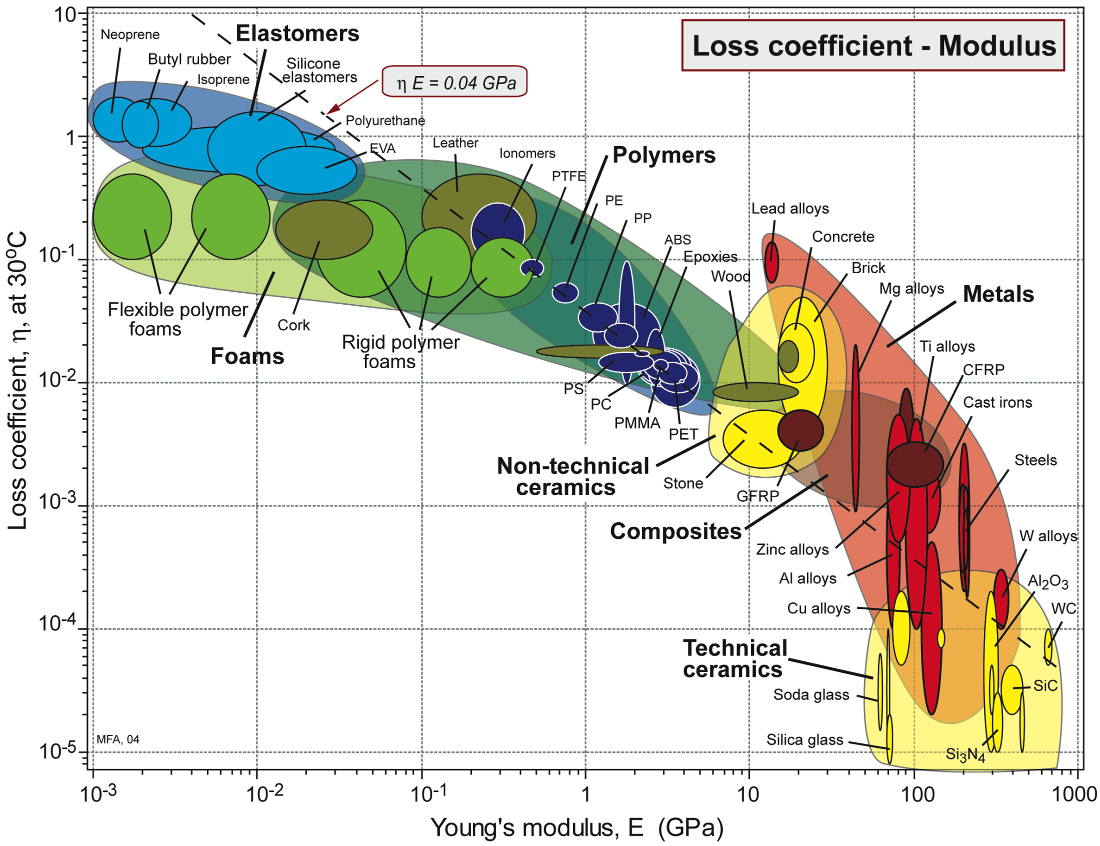
Laser Doppler Vibrometer

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \rightarrow E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



For a complex natural frequency ω:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh} \right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1} .$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12} \left[\frac{h_{31} - d}{1+g} \right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2} (h_{21} - d) + E_3 h_3 (h_{31} - d) \right] \left[\frac{h_{31} - d}{1+g} \right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t \, C \, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.85a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r, θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$ka =$	2.404	3.832	5.135	6.379
$ka =$	5.520	7.016	8.417	9.760
$ka =$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}\ .$$