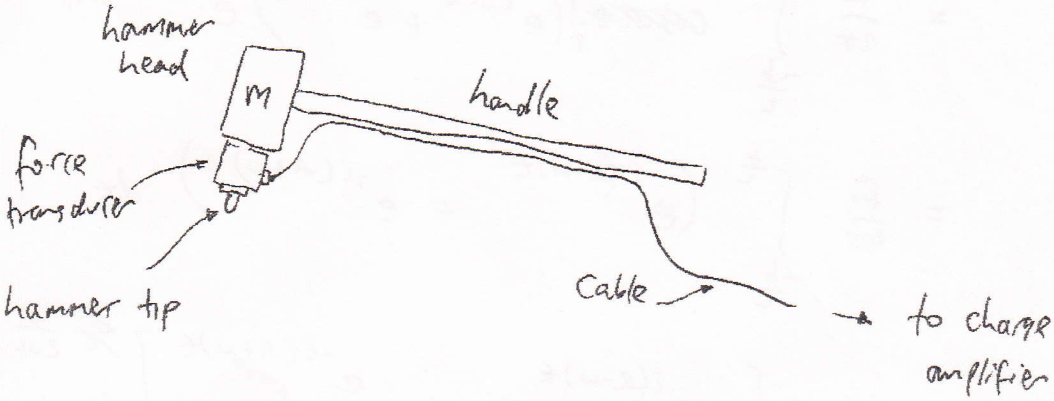
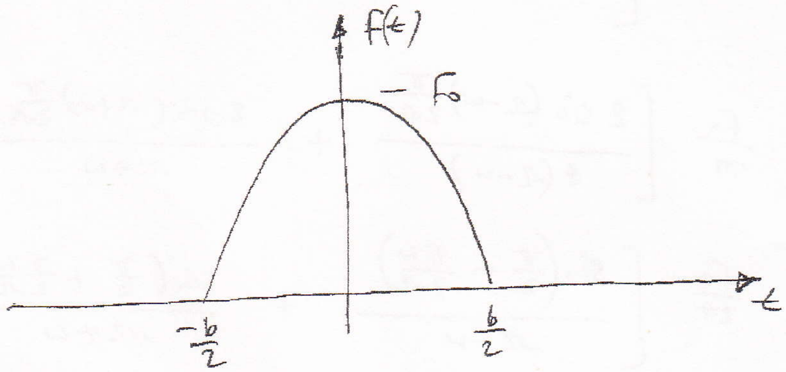
Part IIB 4C6 Solutions

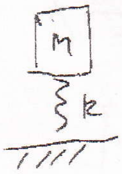
1 (a)



(b) (i)



$$\Omega \frac{b}{2} = \frac{\pi}{2} \quad \therefore \quad \underline{\underline{\Omega = \frac{\pi}{b}}}$$



$\Omega = \sqrt{\frac{k}{M}}$ "natural frequency" of cosine pulse

(ii) $$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{F_0}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \cos \Omega t \; e^{-i\omega t}\, dt$$

$$= \frac{F_0}{2\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \cancel{\cos \Omega t}\, \frac{1}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right) e^{-i\omega t}\, dt$$

$$= \frac{F_0}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(e^{i(\Omega-\omega)t} + e^{-i(\Omega+\omega)t}\right) dt$$

$$= \frac{F_0}{4\pi} \left[\frac{e^{i(\Omega-\omega)t}}{i(\Omega-\omega)} - \frac{e^{-i(\Omega+\omega)t}}{i(\Omega+\omega)} \right]_{\cancel{-\frac{b}{2}}\ -\frac{\pi}{2\Omega}}^{\cancel{\frac{b}{2}}\ \frac{\pi}{2\Omega}}$$

$$= \frac{F_0}{4\pi} \left[\frac{2 \sin (\Omega-\omega)\frac{\pi}{2\Omega}}{\cancel{i}(\Omega-\omega)} + \frac{2 \sin (\Omega+\omega)\frac{\pi}{2\Omega}}{\Omega+\omega} \right]$$

$$= \frac{F_0}{2\pi} \left[\frac{\sin\left(\frac{\pi}{2} - \frac{\pi\omega}{2\Omega}\right)}{\Omega-\omega} + \frac{\sin\left(\frac{\pi}{2} + \frac{\pi\omega}{2\Omega}\right)}{\Omega+\omega} \right]$$

but $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

& $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

$$\therefore F(\omega) = \frac{F_0}{2\pi} \left(\frac{\cos \frac{\pi\omega}{2\Omega}}{\Omega-\omega} + \frac{\cos \frac{\pi\omega}{2\Omega}}{\Omega+\omega} \right)$$

$$= \frac{F_0}{2\pi} \cos \frac{\pi\omega}{2\Omega} \left[\frac{\Omega+\omega + \Omega-\omega}{\Omega^2 - \omega^2} \right]$$

$$= \frac{\Omega F_0}{\pi} \frac{\cos \frac{\pi\omega}{2\Omega}}{\Omega^2 - \omega^2}$$

$$\therefore \quad B = \frac{\Omega F_0}{\pi}$$

Consider $F(\omega)$, $\omega \to 0$ $= \frac{B}{\Omega^2}$

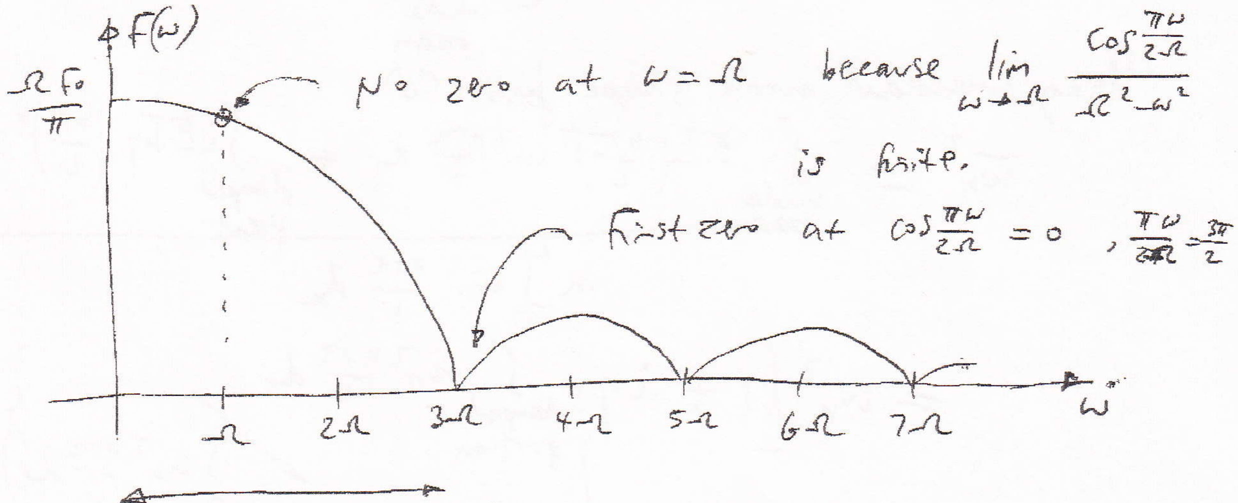
and this is the d.c. component of the impulse

$$\therefore \frac{B}{\Omega^2} = \frac{1}{2\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} F_0 \cos \Omega t \, dt$$

$$= \frac{1}{2\pi}\frac{F_0}{\Omega}\Big[\sin \Omega t\Big]_{-\cancel{\frac{b}{2}}\ -\frac{\pi}{2\Omega}}^{\cancel{\frac{b}{2}}\ \frac{\pi}{2\Omega}}$$

$$= \frac{F_0}{\pi \Omega}$$

$$\therefore B = \frac{\Omega F_0}{\pi}$$



Energy of impulse is here , $\omega < 3\Omega$

$$\therefore 2\pi f < 3\frac{\pi}{b}$$

$$\therefore f < \frac{1.5}{b}$$, roughly as stated.

2 (a) From data sheet, Rayleigh quotient is

$$\omega^2 \simeq \frac{\frac{1}{2}\int EI\, w''^2\, dx}{\frac{1}{2}\int \rho A\, w^2\, dx} \qquad \text{with } \rho A = m$$

Assuming mode $u_n = \sin\frac{n\pi x}{L}$, frequencies will be exact:

$$\omega_n^2 = \frac{EI\left(\frac{n\pi}{L}\right)^4 \cdot L/2}{m \cdot L/2} = \frac{EI}{m}\left(\frac{n\pi}{L}\right)^4$$

(b) If damping is described via complex moduli, replace things like E in expressions for potential energy by $E(1+i\eta)$. Now if damping is light the mode shape should not be changed much from the undamped case. So use undamped mode shape in Rayleigh's principle and obtain an expression for the approximate complex frequency, and hence the modal damping factor

(c) For partial damping the Rayleigh quotient for the beam gives

$$\overline{\omega}_n^2 \simeq \frac{\frac{1}{2}\int_{\substack{\text{undamped}\\\text{portion}}} EI u_n''^2\, dx + \frac{1}{2}\int_{\substack{\text{damped}\\\text{portion}}} E(1+i\eta) I u_n''^2\, dx}{\frac{1}{2} m \int_{\substack{\text{whole}\\\text{beam}}} u_n^2\, dx}$$

Using sinusoidal mode shape from (a):

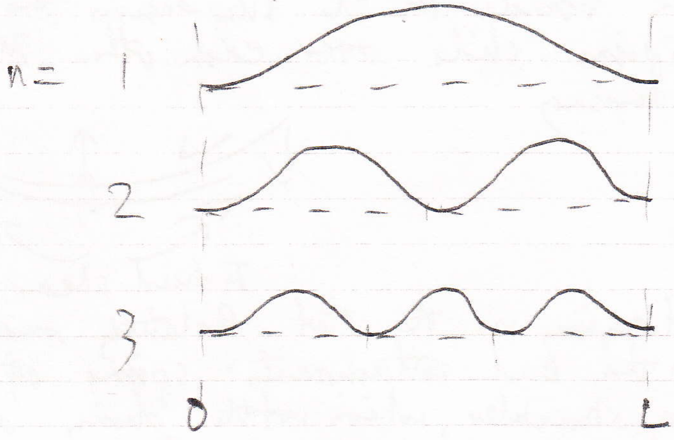
$$\overline{\omega}_n^2 \simeq \frac{\int_{\substack{\text{whole}\\\text{beam}}} EI\left(\frac{n\pi}{L}\right)^4 \sin^2\frac{n\pi x}{L}\, dx + \int_{\substack{\text{damped}\\\text{part}}} iEI\eta\left(\frac{n\pi}{L}\right)^4 \sin^2\frac{n\pi x}{L}\, dx}{m\int_0^L \sin^2\frac{n\pi x}{L}\, dx}$$

$$= \omega_n^2\left[1 + i\eta\left\{\int_{\substack{\text{damped}\\\text{portion}}} \sin^2\frac{n\pi x}{L}\, dx \Big/ \int_0^L \sin^2\frac{n\pi x}{L}\, dx\right\}\right]$$

So

$$\eta_n = \frac{2\eta}{L}\int_{\substack{\text{damped}\\\text{portion}}} \sin^2\frac{n\pi x}{L}\, dx$$

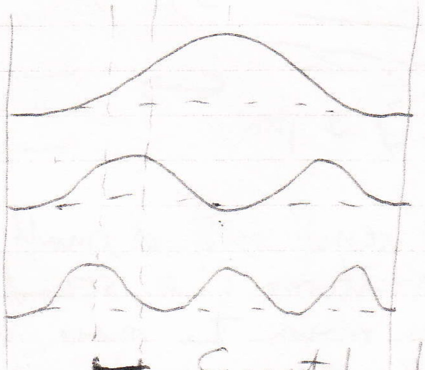
2 (d) Plot $\sin^2 \frac{n\pi x}{L}$ for the first 3 modes:



A patch of length $L/10$ can be chosen at any position, and modal damping is proportional to the area of the curves in this patch.

So for example a patch placed centrally will have a large effect on modes 1 and 3, but little effect on mode 2.

To have a good effect on all three, need to choose a region where all curves are reasonably high. A good place might be here:



Suggested damping region

3 (a) 1: Gross sliding. In response to large-amplitude motion of the vehicle on its suspension the layers of the leaf spring slide over each other, introducing frictional damping

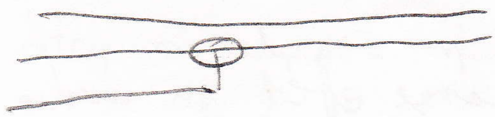
Induced shear like this

2: Material damping in the steel. Relatively small effect

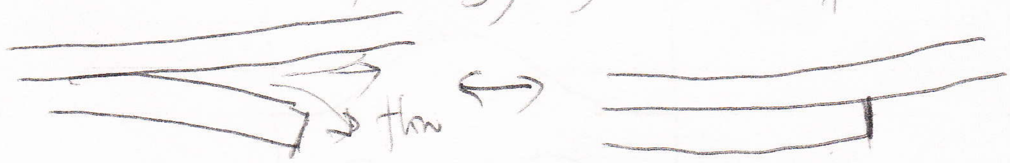
3: Friction in the end attachments: spring often attached to chassis via shackles, which rotate during dynamic motion

4: Micro-slip at ends of leaf springs. For small-amplitude vibration at higher frequencies may not get gross sliding but will still have micro-slip around the stress ~~concentrations~~:

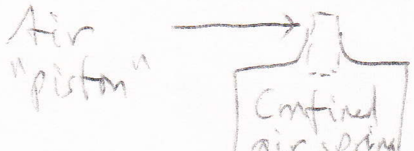
During bending, need a singularity in tangential stress to prevent slip, so some always occurs



5: Air pumping. Small gaps between the leaves of the spring may open and close, forcing air in and out and losing energy by viscous effects



(b) (i) If a rigid vessel with a small opening is exposed to pressure variations (i.e. sound waves) a resonance can occur in which the mass of a "plug of air" in the neck bounces on a spring resulting from the air trapped in the volume



The mass of the "air piston" is determined by the length and the cross-sectional area of the neck, together with the density of air.
The spring stiffness is determined by the volume of the vessel, and the compressibility of air. The air density and compressibility effects combine to depend only on the speed of sound.

3 (b(i)) contd.

In a Helmholtz resonator, the pressure within the vessel is approximately uniform at all points — so the shape of the vessel doesn't matter. But if the longest dimension of the vessel is long enough that at the nominal Helmholtz resonance frequency a significant fraction of a wavelength of sound can fit into the vessel, this approximation becomes poor.

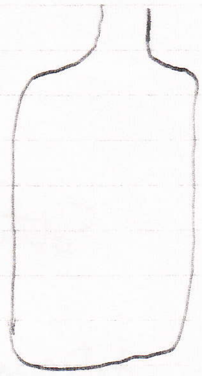
An organ pipe resonance is at the opposite extreme: the pressure varies so that a quarter wave or a half wave (depending on the boundary conditions) fits into the length of the tube. The dimension of the tube is then crucial to the frequency, unlike a Helmholtz resonator.

End correction: the Helmholtz "air piston" is really representing the kinetic energy of the air flow through the neck. This high-speed flow is not confined to the geometric length of the neck, it extends a bit on both sides. This is captured in the formula by an end correction to be added to the geometric length when calculating the effective mass and hence the resonance frequency.

(c) Drink bottle has volume $2L = 2\times10^{-3}\ m^3$

Neck has area approx πa^2 with a radius $1cm = 10^{-2}\ m$

$\therefore\ S \simeq 3\times10^{-4}\ m^2$



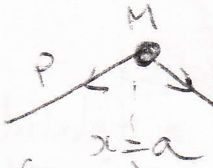
Neck length is approx 2cm, plus an "unflanged" end correction outside, and a "flanged" end correction inside

So $L \simeq 2 + 0.6\times1 + 0.8\times1$ cm
$= 3.4cm = 0.034m$

So frequency $\omega = c\sqrt{\frac{S}{VL}} \simeq 340\times\sqrt{\frac{3\times10^{-4}}{6.8\times10^{-5}}}$ rad/sec

$= 714$ rad/sec $\rightarrow 114\ Hz$

4 (a) Near the mass M:



Newton II $\rightarrow$

$$M\frac{\partial^2 w}{\partial t^2}(a,t) = P\left[\frac{\partial w}{\partial x}(a+,t) - \frac{\partial w}{\partial x}(a-,t)\right]$$

Also need $w(a-,t) = w(a+,t)$

(b) Governing equation $P\dfrac{\partial^2 w}{\partial x^2} = m\dfrac{\partial^2 w}{\partial t^2}$

For a mode, let $w = u(x)e^{i\omega t}$

Then $Pu'' = -m\omega^2 u$, with general solution

$$u = A\cos kx + B\sin kx, \quad k^2 = \frac{m\omega^2}{P}, \; A, B \text{ constants.}$$

Solution has to look like this both sides of the mass M.

So let $u = \begin{cases} K_1 \sin k(x+L) & -L \le x \le 0 \\ K_2 \sin k(L-x) & 0 \le x \le L \end{cases}$

to satisfy $u=0$ at $x=-L$, $x=L$.

For symmetric modes, $K_1 = K_2$, for antisymmetric modes $K_1 = -K_2$

<u>Symmetric case</u>: result from part (a) requires at $x=0$

$$-M\omega^2 \sin kL = 2P(-k\cos kL)$$

$$\therefore \tan kL = \frac{2Pk}{M\omega^2} = \frac{2P}{M}\frac{k}{(Pk^2/m)} = \frac{2m}{Mk}$$

Solutions for k then give $\omega = k\sqrt{\dfrac{P}{m}}$

<u>Antisymmetric case</u>: must have $u(0) = 0$

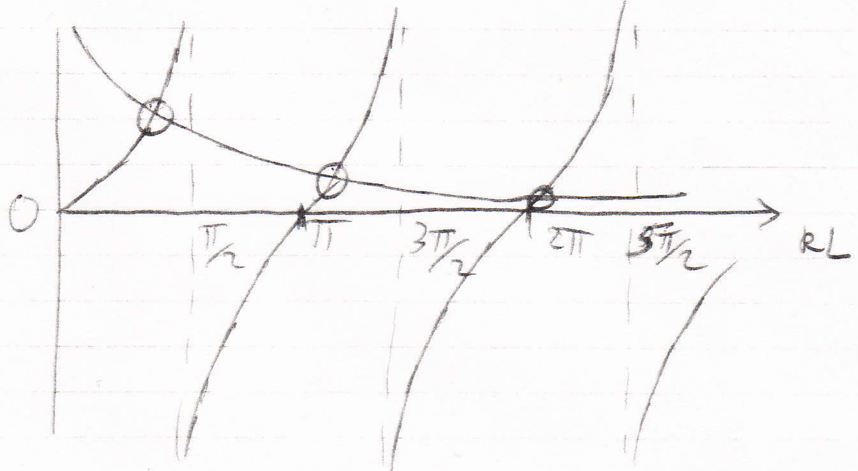
$\therefore \sin kL = 0$, so $kL = n\pi$, $n = 1, 2, 3 \ldots$

Then $\omega = \dfrac{n\pi}{L}\sqrt{\dfrac{P}{m}}$, $n = 1, 2, 3 \ldots$

4 (c) If the mass M is ~~fixed~~, the frequencies are those of two separate strings of length L. These are of course identical, for each mode of the strings.

Now relax the constraint and allow M to move. New frequencies must interlace. So there must still be one frequency equal to those of the decoupled strings: these are the antisymmetric frequencies from (b).

The other frequencies, the symmetric modes must move downwards, but can't move further than the next mode down in order to interlace. Check by graphical solution for symmetric mode equation: plot $\tan kL$ and $\frac{2m}{Mk}$ against k and see where they cross



Roots always lie between $kL=n\pi$ and $(n+\frac{1}{2})\pi$, $n=0,1,2,\ldots$

so interlacing is confirmed. $M\to\infty$ gives $n\pi$, $M\to 0$ gives $(n+\frac{1}{2})\pi$

(d) For mass in a general position, the interlacing argument still applies. $M\to\infty$ decouples the two strings. Their separate frequencies can easily be written down, as for the example done in the lectures.

Now as the mass becomes finite, the new frequencies interlace, and move downwards except for special cases similar to the one in part (c). If certain frequencies of the 2 separate strings are identical, that frequency must still be present with finite M, whatever the value of M.