1

				velocity			displacement	
Mode	f_n	Δf	$Q = \frac{f_n}{\Delta f}$	a_n	$Q a_n$	$20\log_{10}(Q a_n)$	$d_n = \frac{a_n}{\omega_n}$	$20\log_{10}(Q d_n)$
1	0.2	0.02	10	0.8	8	18	0.637	16
2	0.6	0.03	20	0.25	5	14	0.0663	2
3	0.65	0.0013	500	0.25	125	42	0.0612	30
4	1.2	0.012	100	0.1	10	20	0.0133	2
	(Hz)	(Hz)		(mms^{-1}/N)		dB	mm/N	dB

"point receptance" ∴ expect anti resonance between each peak

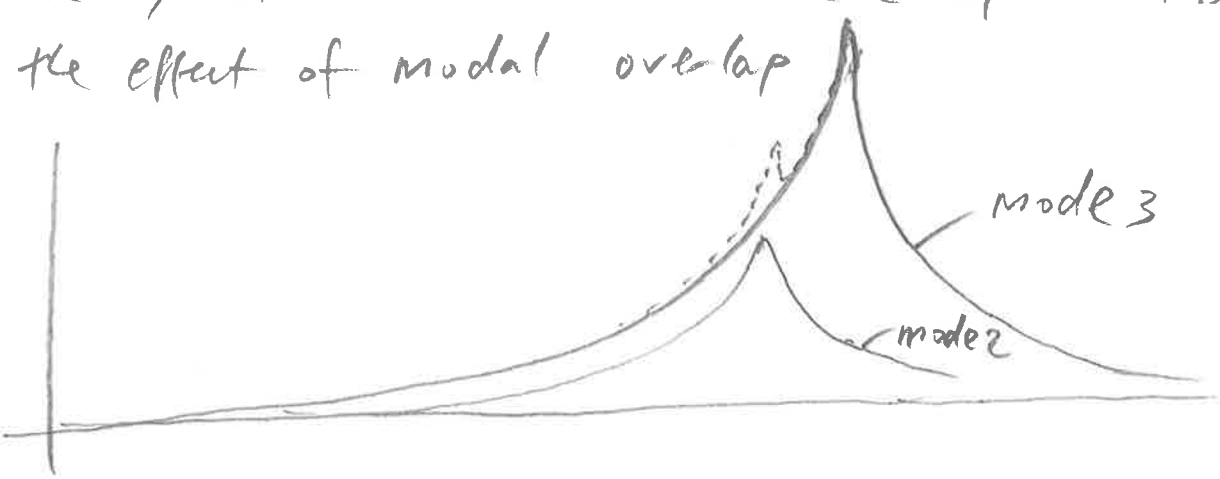
not really necessary, just plot velocity transfer function

Near zero frequency, note free edges ∴ rigid body mode ∴ $\left|\frac{U}{F}\right| = \frac{1}{m\omega}$ so tends to $+\infty$ as $\omega \to 0$

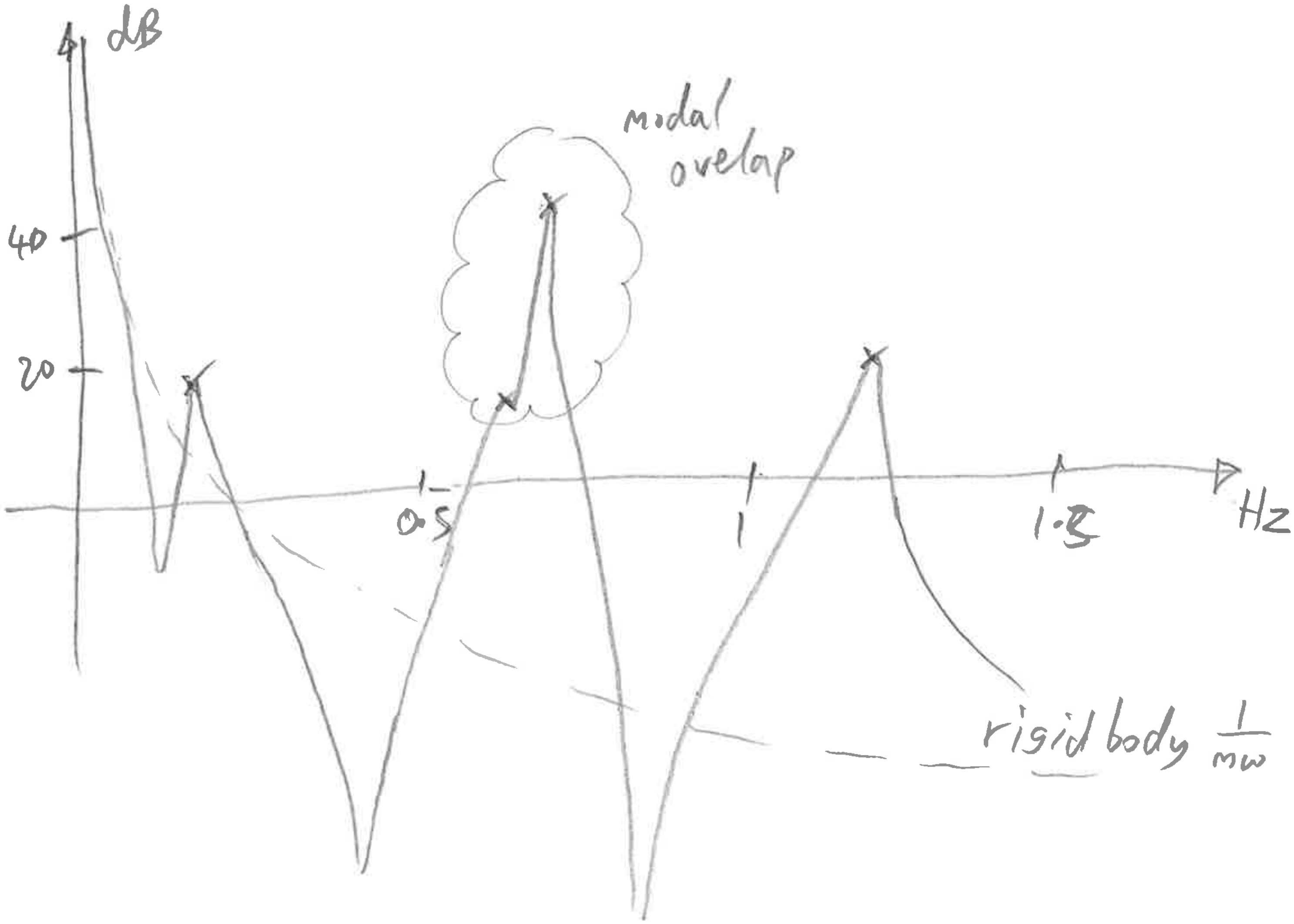
(or if plotting displacement then $\left|\frac{x}{F}\right| = \frac{1}{m\omega^2}$)

Modal overlap for modes 2 & 3

Because mode 3 is so big (25 times taller than mode 2) it will swallow mode 2 up — this is the effect of modal overlap

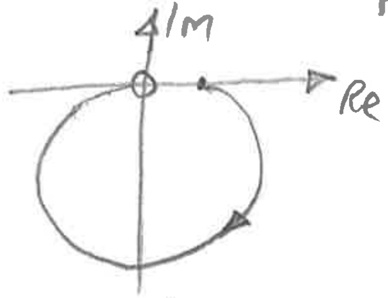


plot velocity (or displacement) tfr

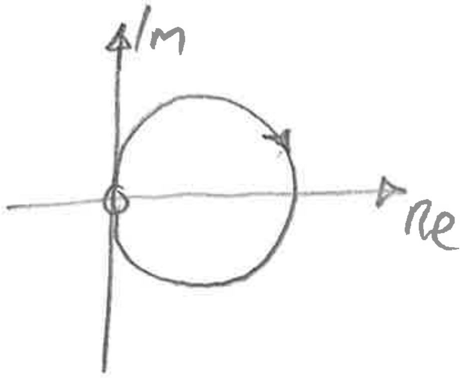


For modal circles, each mode does $\frac{w}{F} = \frac{u_n(x)\,u_n(y)}{\omega_n^2 + 2i\zeta_n\omega\omega_n - \omega^2}$

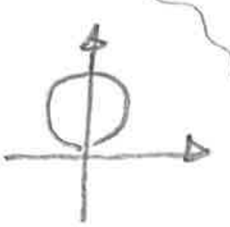
for displacement

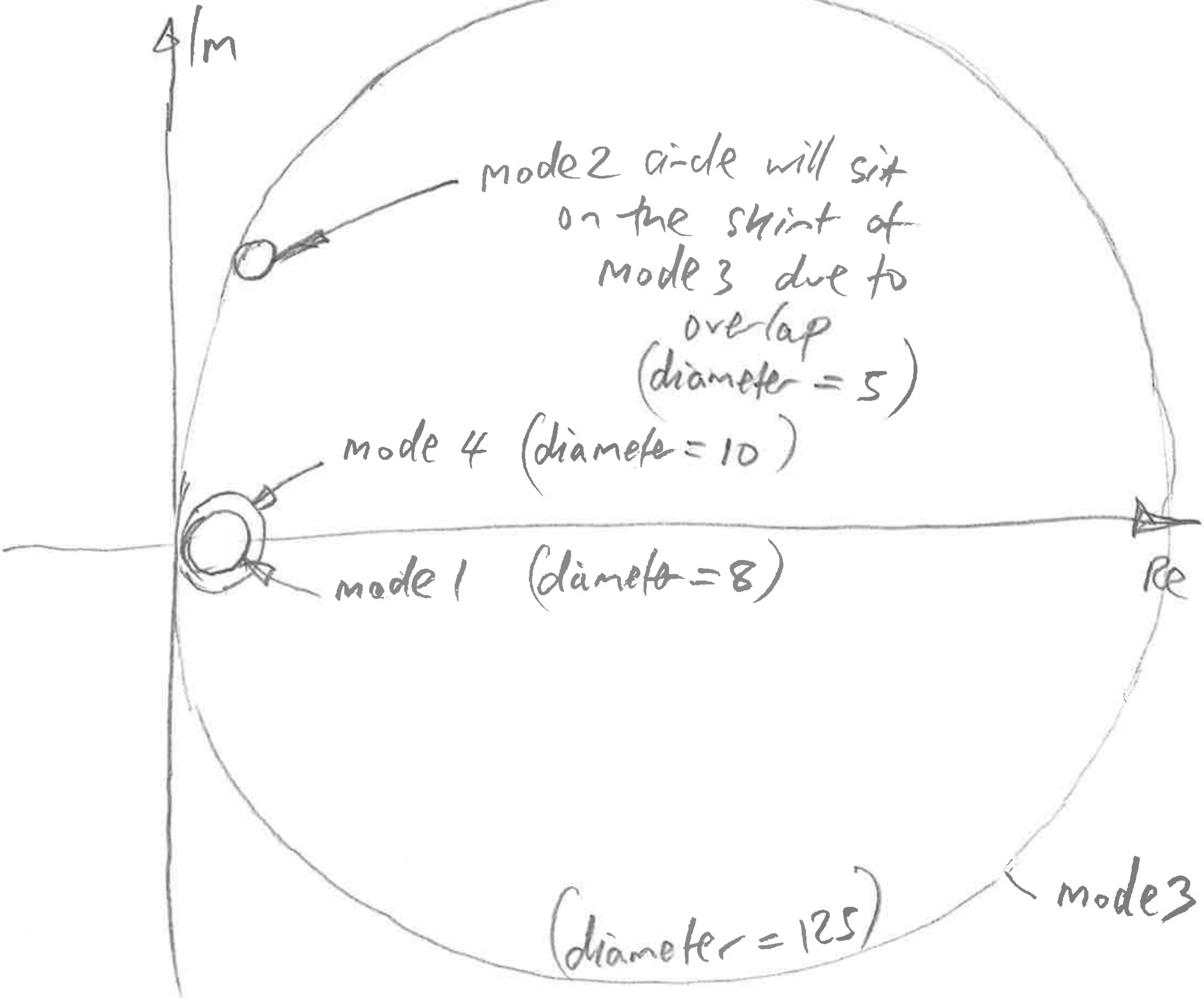


for velocity
multiply by $i\omega$

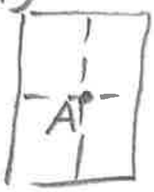


note, this is acceleration



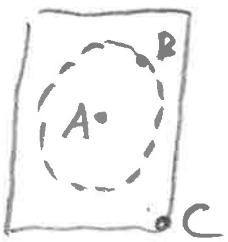


(c) - frequencies and half power bandwidth values will be unchanged, modal amplitudes will vary



- modes with nodal lines passing through the centre may have been missed at A

- Modes detected at the centre may be lost at B



- modal amplitudes can be negative say between A and C

2 (a) Damping of engineering structures is commonly modified by adding damping layers. A free-layer treatment consists of a layer of viscoelastic material applied to the original structure. A constrained layer has an additional constraining layer on top of the damping layer. In a free layer treatment, bending vibration of the underlying structure results in in-plane stretching and compression of the damping layer. A proportion of the strain energy is carried by this layer, and if a high-damping material is chosen then a significant proportion of that energy is dissipated in each cycle of vibration. By adding a constraining layer to form a three-layer structure, usually of thin but stiff material, additional shear deformation can be induced in the damping layer. This can allow higher levels of strain for a given vibration level in the underlying structure, and hence gives the possibility of a higher rate of energy dissipation.

The free layer has the advantage of being cheap and simple to apply: it can be painted or sprayed onto the structure, and no problems arise from geometric complications of the structure. The same material may also provide other functions, such as thermal insulation and corrosion protection. The main disadvantage is that in-plane strains induced by bending vibration are usually quite small so that levels of energy dissipation may be rather low unless the added layer is very substantial, which has implications for cost, weight and space.

A constrained layer can give higher performance, although this is by no means guaranteed as the numerical example from the lecture notes demonstrates. The shear mechanism is more efficient at shorter wavelengths, so constrained layers tend to be more effective at higher frequencies. The system is more complicated than a free layer, so is likely to be more expensive and, if built up in situ as part of the fabrication, requires careful manufacture to ensure conformity between the layers, especially if the structure geometry is complicated. Most commonly, constrained layers are applied in the form of pre-prepared tape with the damping and constraining layers both present. Such tape is commercially available in a wide variety of grades targeted at different applications and frequency ranges. It helps with the problem of conformity, but the installation has to be carefully controlled to avoid wrinkles in the constraining layer since high in-plane stiffness in that layer is necessary for effective function.

(b) The Data Sheet gives the combined effective stiffness of a 2-layer beam representing the simplest model of a free-layer treatment applied to a plate. The correspondence principle states that the viscoelastic beam problem can be addressed by solving the undamped problem, and then using suitable complex values for elastic moduli. In this case, if the original structure is assumed to be undamped, this amounts to replacing E_2 by $E_2(1+i\eta)$, and hence e by $e(1+i\eta)$. The result is a complex value of the effective EI. According to Rayleigh's principle, the squared frequency of any mode of a beam is proportional to the effective EI, and this is the only place that complex values can enter the problem. Hence using the result on P7 of the Data Sheet, the Q factor of a typical mode satisfies

$$Q \approx \frac{\mathrm{Re}(EI)}{\mathrm{Im}(EI)}$$

But from P8 of the Data Sheet,

$$EI = E_1 I_1\left[1+eh^3+3(1+h)^2 eh/(1+eh)\right]$$

and if we assume $eh \ll 1$ (added layer has only a small influence) then

$$EI \approx E_1 I_1\left[1+eh^3+3(1+h)^2 eh\right]$$

2 Cont. so that $\mathrm{Re}(EI) \approx E_1 I_1$ while $\mathrm{Im}(EI) \approx E_1 I_1 e\eta\left[h^3 + 3h(1+h)^2\right]$

and so

$$Q \approx \frac{1}{eh\eta\left[4h^2 + 6h + 3\right]}$$

(c) For a free layer of given thickness and a given underlying structure, part (b) shows that

$$Q \propto \frac{1}{e\eta} \propto \frac{1}{E_2\eta}$$

So two materials will have the same damping performance if they have the same value of $E_2\eta$. Given the log-log axes of the design chart, this means that equivalent materials lie along straight lines with slope -1: as indicated by the dashed line included on the chart. Higher damping is achieved by pushing the line higher in the diagram, lower damping when it is lower. What the chart shows is that although the separate properties vary over a very wide range, there is a strong correlation approximately parallel to the design line, so that differences between materials are unexpectedly small.

Along the particular line shown on the chart, examples of equivalent materials could be leather, PTFE and Aluminium oxide. However, this is misleading. To justify the approximation that the added layer does not change the underlying mechanical behaviour much, we require a Young's modulus much lower than that of steel, of which the structure is made. So materials of interest might be well to the left in the chart. The chart suggests diminishing returns for materials with very low E at the left. Taking the criterion at face value, the best material might be lead. This is indeed a material used in the past for this purpose, but it now has problems of cost and toxicity and would not be chosen. Something towards the top edge of the "polymers" bubble might be a good compromise.

3 (a) Substitute $p = f(x,y)\, e^{i(\omega t - \lambda z)}$
to find $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} - \lambda^2 f = -\omega^2 c^{-2} f$

This is the same as $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = -\dfrac{m}{T}\Omega^2 w$
for the membrane at frequency Ω (data sheet)
if $\omega^2 c^{-2} - k^2 = \dfrac{m}{T}\Omega^2$

On the boundary, if $\underline{n}$ is the outward-pointing normal vector, the condition is $\underline{n}\cdot\nabla p = 0$
So for a circular duct, if f is written as a function of polar coördinates r, θ, we need $\dfrac{\partial f}{\partial r} = 0$ on $r = a$

(b) From Data Sheet, possible solutions are
$$f(r,\theta) = \begin{matrix}\sin\\ \cos\end{matrix}\Big\} n\theta \; J_n(kr)$$
where $k^2 = \Omega^2 \dfrac{m}{T}$ for membrane
$= \omega^2/c^2 - \lambda^2$ for the duct

Boundary condition is $J_n'(ka) = 0$ — determines certain values of k.
For a given value, $\lambda^2 = \omega^2/c^2 - k^2$
For a propagating wave need λ to be real,
so $\lambda^2 \geq 0$, so $\omega^2/c^2 \geq k^2$
If $\omega^2/c^2 < k^2$, $\lambda^2 < 0$ so λ is imaginary, and $e^{i\lambda z}$ describes decaying (evanescent) disturbances. So each value of k allows a wave to propagate for $\omega^2 \geq k^2 c^2$

3 (c) From the graphs, values where $J_n'(z) = 0$ are roughly:

$n = 0 \quad z = 0, 4, 7$
$n = 1 \quad z = 2, 6$
$n = 2 \quad z = 3, 7$
$n = 3 \quad z = 4, 8$

Then $ka = z$, and critical $\omega = kc = \frac{zc}{a}$

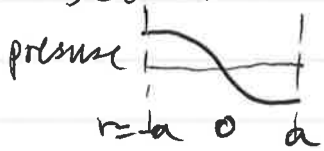
So frequency in Hz is $\frac{zc}{2\pi a} = \frac{340}{\pi} z = 108 z$

So the first few frequencies are:

1. $n = 0$, $f_c = 0$
2. $n = 1$ $f_c \simeq 216$ Hz
3. $n = 2$ $f_c \simeq 324$ Hz
4. $n = 0$ $f_c \simeq 432$ Hz
5. $n = 3$ $f_c \simeq 432$ Hz
6. $n = 1$ $f_c \simeq 648$ Hz

(1) is a plane wave: uniform pressure in cross section

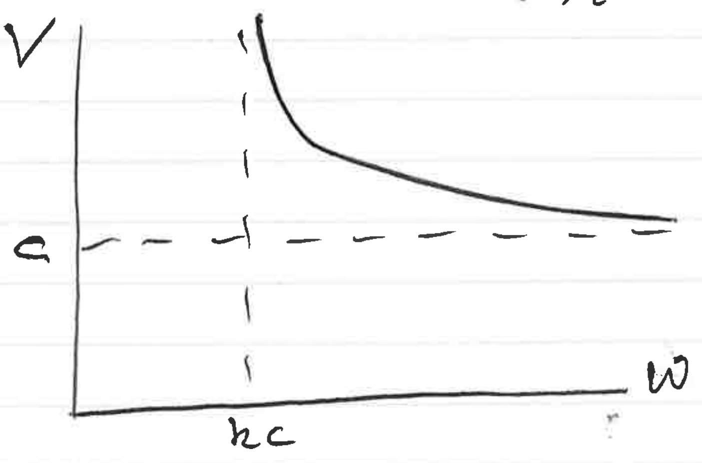
(2) has one nodal diameter 



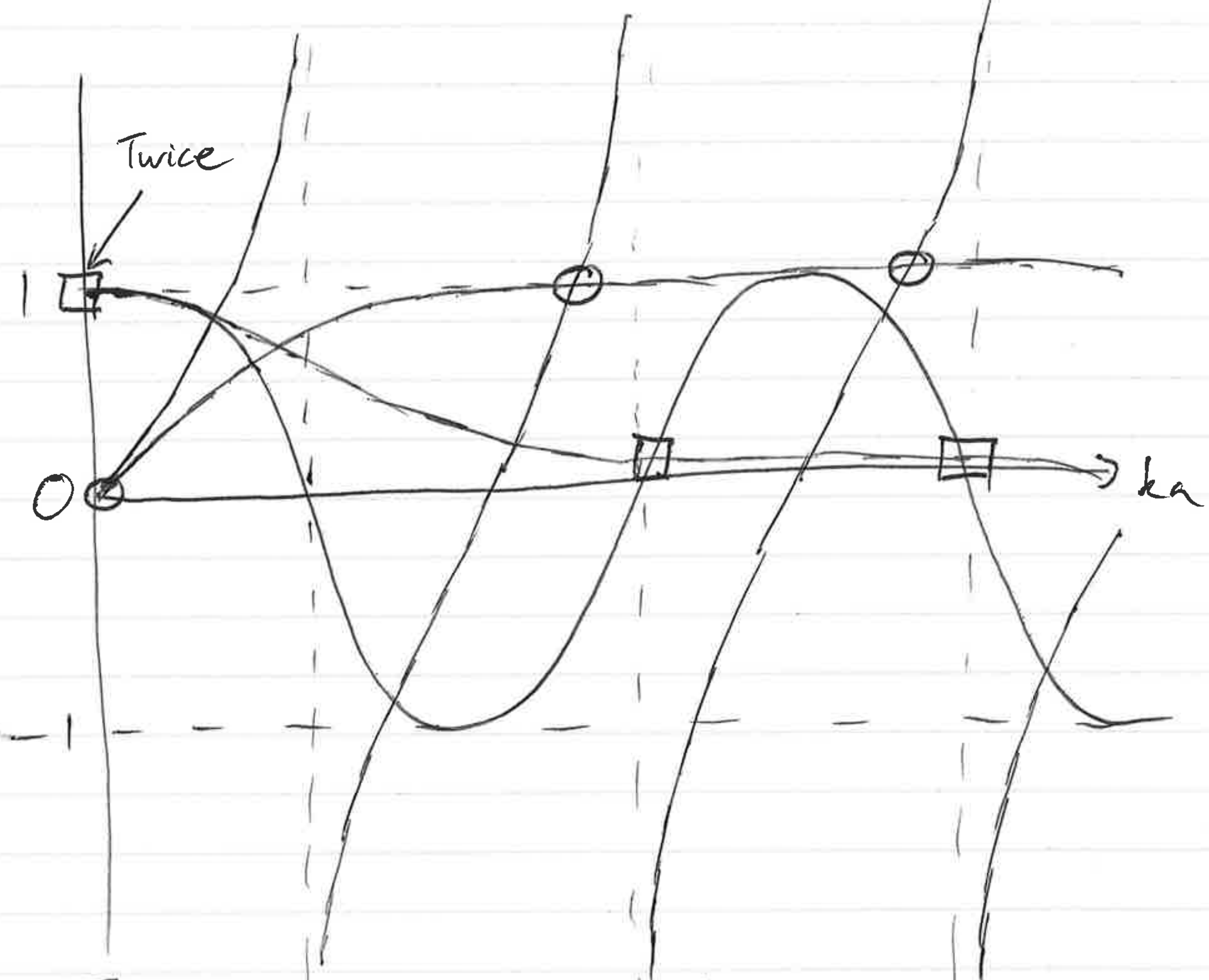
(3) has two nodal diameters 

5 Modes can propagate at 500 Hz.

(d) Speed of wave $V = \frac{\omega}{\lambda} = \frac{\omega}{\sqrt{\omega^2/c^2 - k^2}}$



4 (a) In going from free-free to pinned-free, just one constraint has been added to the beam (zero motion at one end) so the conditions of the theorem are satisfied. The pinned-free beam has the extra constraint, so its frequencies must lie in the gaps from the free-free beam.
Note that the free-free beam has 2 DoF in rigid-body motion at $\omega = 0$, while the pinned-free beam has only one.
Now plot $\cos \alpha L$, $\frac{1}{\cosh \alpha L}$, $\tan \alpha L$ and $\tanh \alpha L$ to see where the two sets of frequencies fall: note that $\omega \propto \alpha^2$, so ω's interlace if α's interlace.



□ free-free ○ pinned-free
Interlacing can be clearly seen

4 Cont.

(b) (i) A single spot weld treated as a pinned constraint satisfies the conditions of the interlacing theorem: if it comes apart, one constraint is released, and the natural frequencies of the panel will move so that one frequency of the welded panel lies in every space between frequencies of the unwelded panel.
(ii) If the weld acts as a clamped constraint between the two panels, then it imposes a total of three constraints: equal relative displacement on the two panels, and equal slopes in two perpendicular directions. If these three constraints could be released one at a time, interlacing behaviour would be seen between each pair of successive stages. However, there is no guarantee of interlacing between the initial and final states. It CAN be guaranteed that the lowest frequency is lower than the original, because it must move down (or at least not move up) as each constraint is removed.
(c) (i) Adding a point mass can be regarded as a point coupling between the original guitar and a system with a single resonance at zero frequency (the unattached mass). So the new frequencies of the guitar must interlace with those of the original guitar, plus an extra frequency at zero. If the support conditions of the guitar do not change, then the number of rigid-body modes at zero frequency will not change (6 if it is entirely free). The addition of the extra zero frequency from the detached mass must mean that the lowest non-zero frequency of the guitar moves downwards, and thus that all the other frequencies move downwards (or at least don't move upwards).
(ii) The cavity of the guitar will have a Helmholtz resonance, modified by flexibility of the walls of the box. Blocking the hole prevents the "air piston" from moving, so it is the equivalent of adding a single constraint to impose zero motion on this piston. So the combined frequencies of guitar body and internal air with the plug in place will interlace with the original frequencies.
(iii) When a string is stopped on a real guitar, it is hard to say exactly what happens to the length of string above the player's finger. However, from the point of view of the playing length of the string, which is the length connected to the guitar body, the effect will be exactly the same as if a point constraint had been imposed at the fret position, dividing that string into two portions just as in the example discussed in the lectures. So the new frequencies of the shorter string, coupled to the guitar body, taken together with the frequencies of the length of string above this point constraint, should interlace with the original frequencies of the guitar with the unstopped string. This will be true even if the actual length of string above the fret is not really free to vibrate because the player's finger is interfering with it. In other words, the new frequencies together with some "virtual frequencies" of the missing section of string, should interlace with the original frequencies.

4C6 – Assessor's comments

Q1 Impulse hammer and modal analysis.
Quite well done. Some candidates were not at all well prepared for this question even though it fits in with past papers. Seven with 18/20 or better so the question was clearly do-able. There was a mistake which no candidate spotted nor were any affected by it – that the 1kg panel mass gives a rigid-body mode which would have swamped the measured modes because the units of measured modes was mm/s. The past-paper version will be changed to 1000kg.

Q2 Free-layer damping
Quite well done, but disappointing lack of practicality in choosing appropriate materials. Many didn't quite understand that this was a surface free layer.
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Q3 Sound waves in a duct
This question looked more complicated than it was. Good efforts by some. Well done by those who figured out the boundary condition correctly.

Q4 Interlacing theorem
Very well done by many, but a lot of poor attempts showing little real understanding of how interlacing theorem works. Some didn't know what a bridge is in a guitar.