1 (a) Substitute assumed solutions into governing equations:

$$TA_1\left(f''g + \frac{1}{r}f'g + \frac{1}{r^2}fg''\right) = -mw^2A_1fg - \lambda fg(A_2 - A_1) \quad -(1)$$

$$TA_2\left(f''g + \frac{1}{r}f'g + \frac{1}{r^2}fg''\right) = -mw^2A_2fg - \lambda fg(A_1 - A_2) \quad -(2)$$

For (1) and (2) to be consistent we must have:

$$-mw^2fg - \lambda fg(A_2 - A_1)/A_1 = -mw^2fg - \lambda fg(A_1 - A_2)/A_2$$

$$\Rightarrow \quad (A_2 - A_1)/A_1 = (A_1 - A_2)/A_2 \Rightarrow A_2^2 - A_1A_2 = A_1^2 - A_1A_2$$

$$\Rightarrow \quad A_2^2 - A_1^2 = 0 \Rightarrow (A_2 + A_1)(A_2 - A_1) = 0$$

$$\Rightarrow \quad \underline{A_1 = A_2} \text{ or } \underline{A_1 = -A_2}$$

$$(1) \times r^2 \times \left(\frac{1}{fg}\right) \Rightarrow \quad \frac{1}{f}(r^2f'' + rf') + r^2\underbrace{\frac{1}{T}\left[mw^2 - \lambda(1 - A_2/A_1)\right]}_{k^2} = \frac{-g''}{g} = n^2$$

$$\Rightarrow \quad g'' + n^2g = 0$$

$$r^2f'' + rf' + (k^2r^2 - n^2)f = 0$$

$$k^2 = \frac{1}{T}\left[mw^2 - \lambda(1 - A_2/A_1)\right] \Rightarrow \underline{k^2 = \frac{mw^2}{T}}$$

$$\text{or } \underline{k^2 = \frac{1}{T}\left[mw^2 - 2\lambda\right]} \quad [40]$$

(b) From the data sheet there is a solution to the equation

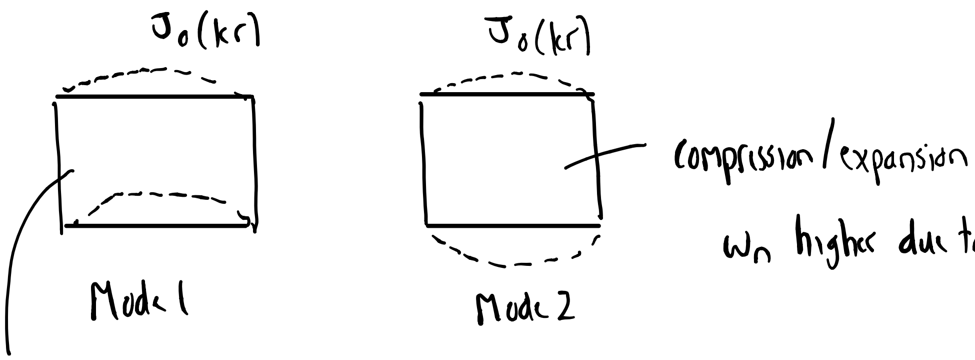
$u(r,\theta) = J_0(kr)$ with $ka = 2.406$

$$\Rightarrow \quad \frac{1}{T}\left[M\omega^2 - \lambda(1 - A_2/A_1)\right] = \left(\frac{2.404}{a}\right)^2$$

$$\Rightarrow \quad \omega^2 = \left(\frac{T}{M}\right)\left(\frac{2.404}{a}\right)^2 + \frac{\lambda}{M}(1 - A_2/A_1)$$

↑ either 0 for $A_2 = A_1$

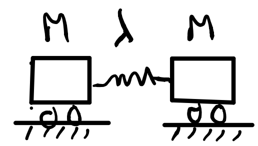
or $\frac{2\lambda}{M}$ for $A_2 = -A_1$



ω_n higher due to added stiffness.

- no compression of air
- same ω_n as single membrane

For $T \to 0$ No membrane stiffness } $\omega_n = 0$ or $\frac{2\lambda}{M}$ analogous to



[40]

(c) The equation becomes $T\left(F''g + \frac{1}{r}F'g + \frac{1}{r^2}Fg''\right) = -\omega^2 mFg + \lambda Fg$

↑ 2λ in second mode replaced by λ

$$\Rightarrow \omega_n^2 = \left(\frac{T}{M}\right)\left(\frac{2.404}{a}\right)^2 + \frac{\lambda}{M}$$

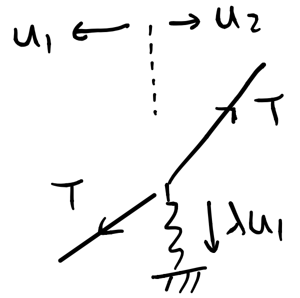
[20]

2. (a) $$T\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^2 u}{\partial t^2} = 0$$

For $u = \begin{cases} A \sin kx \sin \omega t & \longrightarrow \quad u'' = -Ak^2 u \quad \ddot{u} = -A\omega^2 u \\ B \sin k(x-L) \sin \omega t & \longrightarrow \quad u'' = -Bk^2 u \quad \ddot{u} = -B\omega^2 u \end{cases}$

In either case $Tk^2 - m\omega^2 = 0 \Rightarrow \underline{k = \omega\sqrt{\frac{m}{T}}}$ [10]

(b)



$T\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right) = -\lambda u_1$ For equilibrium — (1)

$u_1 = u_2$ For compatibility — (2)

(1) $\Rightarrow$ $Ak\cos k\alpha L - Bk\cos k(\alpha-1)L = \frac{-\lambda}{T} A\sin k\alpha L$

(2) $\Rightarrow$ $A\sin k\alpha L = B\sin k(\alpha-1)L$

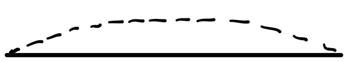
Eliminate B $\Rightarrow$ $Ak\cos k\alpha L \sin k(\alpha-1)L - A\sin k\alpha L \cos k(\alpha-1)L$

$= \frac{-\lambda}{T} A\sin k\alpha L \sin k(\alpha-1)L$

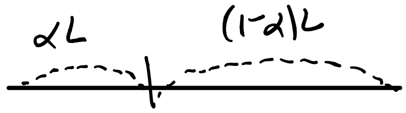
$\Rightarrow \underline{kT\sin kL = -\lambda \sin k\alpha L \sin k(1-\alpha)L}$ [40]

(c) For $\lambda = 0$ $\quad \sin kL = 0 \Rightarrow kL = n\pi \Rightarrow \omega = \sqrt{\frac{T}{m}}\left(\frac{n\pi}{L}\right)$ ✓

For $\lambda = \infty$ $\quad \sin k\alpha L . \sin k(1-\alpha)L = 0 \Rightarrow \omega = \sqrt{\frac{T}{m}}\left(\frac{n\pi}{\alpha L}\right)$ ✓ or $\omega = \sqrt{\frac{T}{m}}\left(\frac{n\pi}{(1-\alpha)L}\right)$ ✓



$\lambda = 0$, simple string

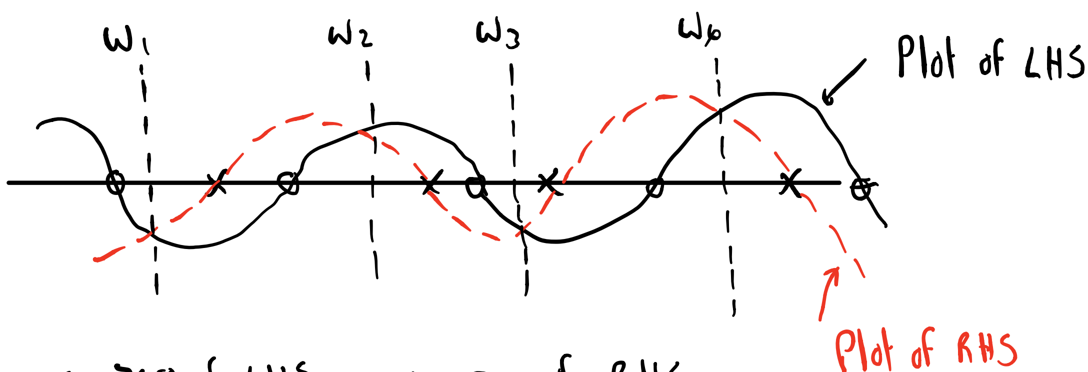


$\lambda = \infty$, two strings

$\Rightarrow$ Results are consistent with physical arguments [15]

(d) $kT \sin kL = -\lambda \sin k\alpha L \sin k(1-\alpha)L$

<u>told</u> that zero's interlace with those of LHS



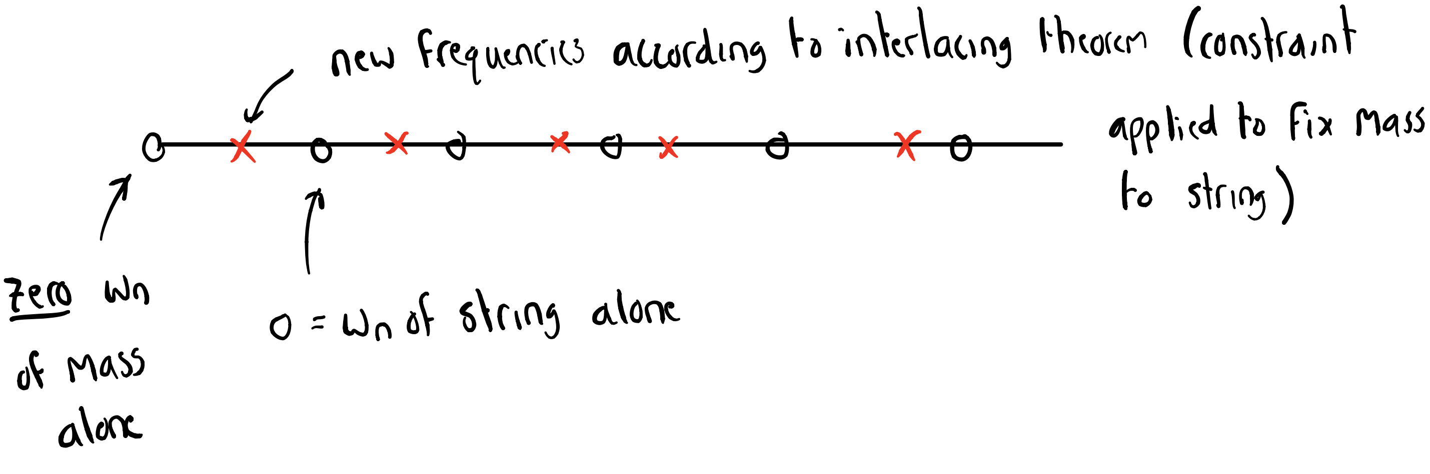
<u>o : zero of LHS</u> <u>x : zero of RHS</u>

At $\omega_1, \omega_2, \omega_3$ then LHS = RHS and frequencies are natural frequencies

One crossing of curves between each pair of roots o—o

⇒ <u>interlacing</u> [20]

(e)



Frequencies interlace as before, but in this case there is a coupled ω_n below ω_1 of the string [15]

3) (a)

$$EI\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} - \frac{mEI}{kAG}\frac{\partial^4 u}{\partial x^2 \partial t^2} = 0$$

$u = \sin px \sin \omega t \Rightarrow$ $\quad p^4 u \quad -\omega^2 u \quad \omega^2 p^2 u$

$$\Rightarrow \quad EIp^4 - m\omega^2 - \left(\frac{mEI}{kAG}\right)\omega^2 p^2 = 0$$

call this α

$$\Rightarrow \quad \underline{\omega^2 = \frac{EIp^4}{m + \alpha p^2}}$$ [10]

(b) When damping is added, $\omega^2 \to \omega^2(1 + i\eta) \Rightarrow i\eta = \frac{\Delta\omega^2}{\omega^2}$

Now $\quad \Delta\omega^2 = \frac{\partial\omega^2}{\partial E}\Delta E + \frac{\partial\omega^2}{\partial G}\Delta G$

$$= \left[\frac{Ip^4}{m + \alpha p^2} - \frac{EIp^4 p^2}{(m + \alpha p^2)^2}\frac{\partial\alpha}{\partial E}\right]\Delta E + \left[\frac{-EIp^4 p^2}{(m + p^2\alpha)^2}\frac{\partial\alpha}{\partial G}\right]\Delta G$$

$\frac{\partial\alpha}{\partial E} \to \frac{\alpha}{E}$, $\quad \frac{\partial\alpha}{\partial G} \to -\frac{\alpha}{G}$

$$\Delta\omega^2 = \left[\frac{EIp^4(m + \alpha p^2) - EIp^6\alpha}{E(m + \alpha p^2)^2}\right]\Delta E + \left[\frac{EIp^6\alpha}{(m + p^2\alpha)^2 G}\right]\Delta G$$

$\Delta E \to iE\eta_E$, $\quad \Delta G \to iG\eta_G$

$$\Delta\omega^2 = \frac{iEIp^4 m\eta_E + iEIp^6\eta_G\alpha}{(m + \alpha p^2)^2}$$

$$\underline{\eta = \frac{\Delta\omega^2}{i\omega^2} = \frac{m\eta_E + \alpha p^2\eta_G}{m + \alpha p^2}}$$ [40]

(c) For equal contribution from shear and bending:

$$m\eta_E = \alpha p^2 \eta_G \Rightarrow p^2 = \frac{m}{\alpha}$$

$$\Rightarrow p^2 = \frac{kAG}{EI} \quad \text{with } A = h^2 \text{ and } I = \frac{1}{12}h^4,\ k = 5/6$$

$$\text{and } G = E/2(1+\nu)$$

$$\Rightarrow p^2 = \left(\frac{5}{6}\right)\left(\frac{12h^2}{h^4}\right)\left(\frac{1}{2(1+\nu)}\right) = \frac{3.84}{h^2}$$

Now $p = \frac{2\pi}{L}$, L = wavelength $\Rightarrow L^2 = (2\pi h)^2\left(\frac{1}{3.84}\right)$

$\Rightarrow \underline{L = 3.2h}$ [25]

(d) $$\eta = \frac{m\eta_E + \alpha p^2 \eta_G}{m + \alpha p^2}$$

Consider $$\frac{\partial \eta}{\partial p^2} = \frac{\alpha \eta_G}{m + \alpha p^2} - \frac{(m\eta_E + \alpha p^2 \eta_G)\alpha}{(m + \alpha p^2)^2}$$

$$= \frac{\alpha \eta_G (m + \alpha p^2) - (m\eta_E \alpha + \alpha^2 p^2 \eta_G)}{(m + \alpha p^2)^2} = \frac{\alpha m(\eta_G - \eta_E)}{(m + \alpha p^2)^2}$$

$\Rightarrow \frac{\partial \eta}{\partial p}$ is always $\underline{+\text{ve for } \eta_G > \eta_E}$ and always $\underline{-\text{ve for } \eta_G < \eta_E}$ [25]

Q4

(a) (i)

Shaker	Hammer
Widely controllable (wide bandwidth, different input signals possible, e.g. swept sine, chirp, etc.)	Governed by mechanical design (bandwidth determined by tip mass and stiffness)
May apply other forces/moments and/or add damping	Applies normal force only
Force magnitude is easily controlled (can be increased in noisy environments)	Hard to achieve large amplitude (signal-noise ratio can be low)
Needs a fixed reference	Easily portable

(ii)

Accelerometer	Vibrometer
Mass loads structure and can add damping via signal wire	Non-contact and remote operation
Sensitivity and bandwidth limited by size (high sensitivity requires large, low-bandwidth device)	Wide range of available settings
Cheap (especially MEMS devices)	Expensive
Easily portable	Needs a fixed reference (vibration of mounting can distort measurement)

(iii). Sensible checks include:

(a) testing the sensor arrangement by sampling the accelerometer output whilst either shaking it on a proprietary 'calibrator' or inverting it by hand – the output should swing approximately from plus g to minus g;

(b) conducting a reciprocity test by comparing measurements of the same transfer function before and after swapping the exciter and sensor positions – the results should be identical by the reciprocal theorem.

4C6 2021 crib supplement

Q4(a)

The general pros and cons are given in the table, and half marks are allocated to these general principles

But for full marks the particular application needs to be considered. Some thoughts:
- that the painting is valuable so great care must be taken to avoid damage
- that non-contact excitation and measurement are therefore preferred, especially on the canvas
- that the frame is heavy/rigid and the canvas is light/flexible
- that access to the painting might be limited and constrained by where it can be moved to
- that the arrangements for packing the painting in a crate might affect the vibration characteristics
- that museum curators are ultra-protective of their exhibits and they don't trust anyone!

Given this background:

(i) Hammer vs Shaker

It's probably OK to use a hammer on the frame (on the back where any tap marks won't be seen). A hammer won't work on the canvas because it is so light-weight. Excitation of the canvas is probably best done by exciting the frame. In any case, reciprocity can be used to work out what the effect would be of excitation of the canvas.

A problem with the hammer approach is that a large hit will be required to get the canvas to move appreciably so the use of a shaker on the frame is probably necessary. The advantage here is that you can tune the frequency to the particular modes that excite the canvas and thereby limit the total vibrational energy being delivered.

And the curator will probably be happier that your not hitting his painting with a hammer.

(ii) Accelerometer vs laser vibrometer

On the canvas there's no choice – must be non-contact so use laser. Also, canvas is light and thin so added mass of accelerometer is undesirable.

On the frame, laser is good, but you might want to use the laser (if you only have one) for the canvas and use several accelerometers on the back of the frame. This of course depends on the curator giving you permission. The argument will be won if you suggest that they pay for a second laser vibrometer. Scanning laser vibrometers can solve the problem – they can do frame and cavas. But they are expensive.

(iii) Checking results

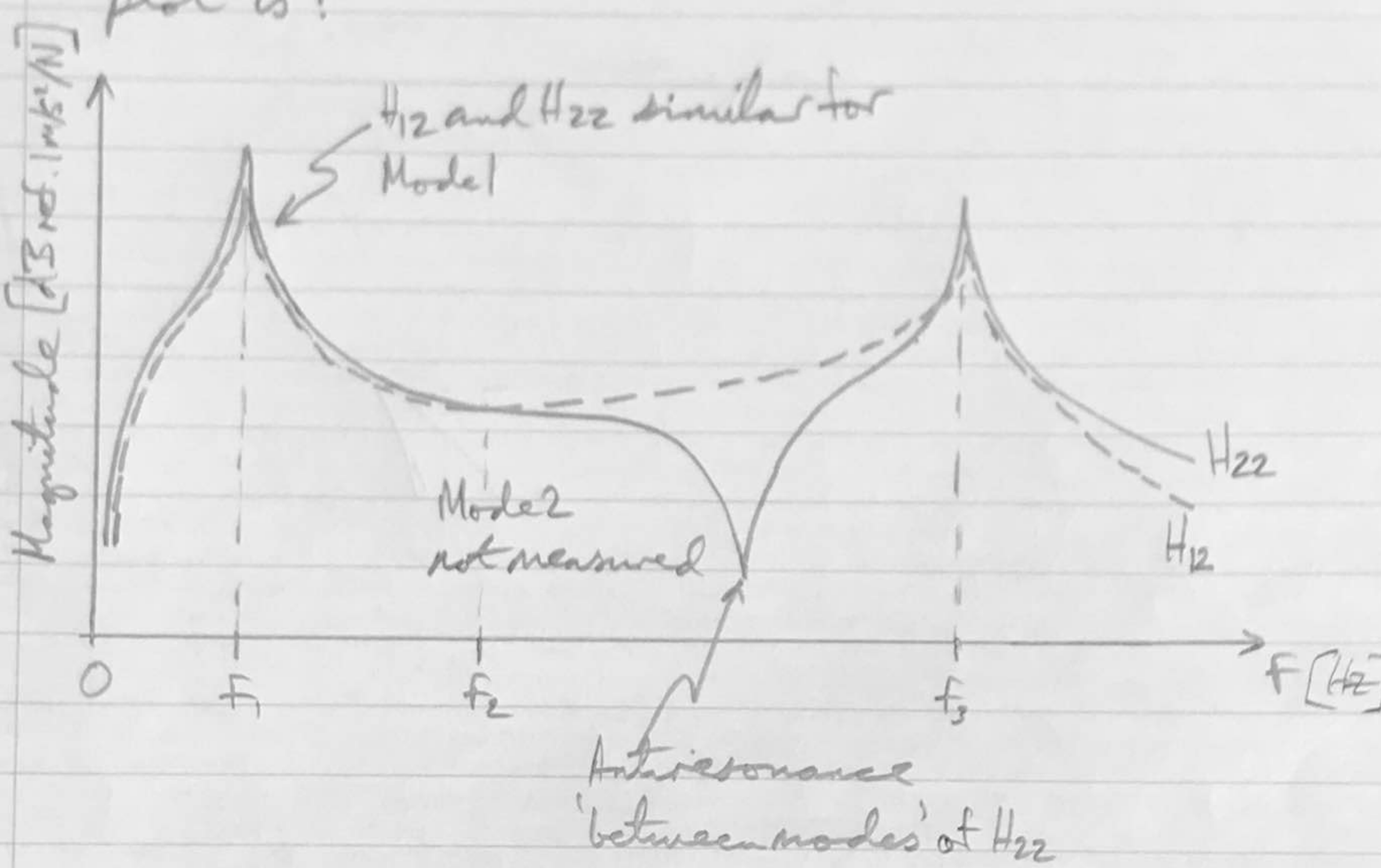
If you're using accelerometers and laser then they can be cross-checked. And reciprocity checks are always very illuminating.

Candidates raising even a few of these points will get good marks.

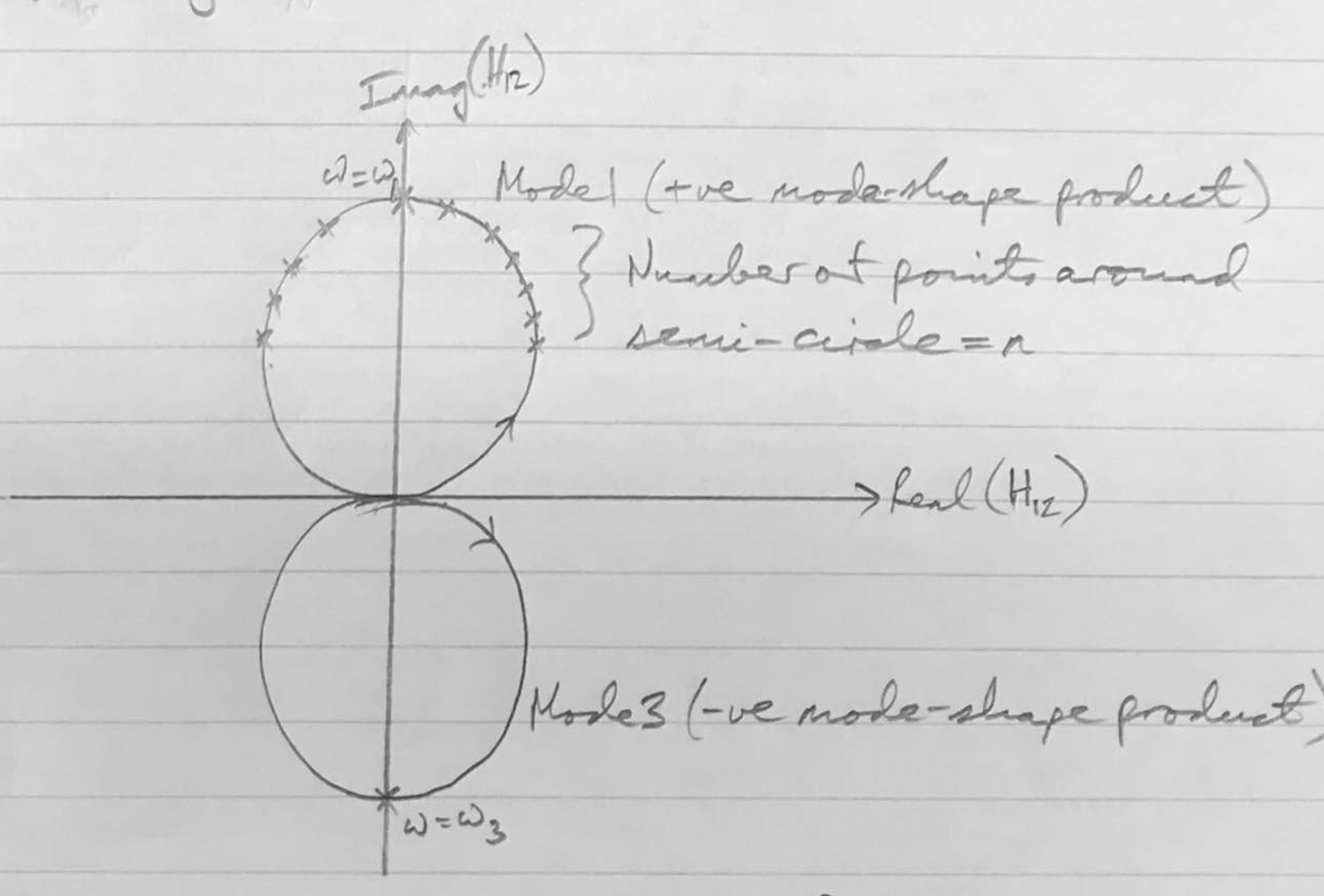
Q4 (continued)

(b)

(i) Point 2 coincides with a node of the second mode (simply-supported beam) so this is never measured. H12 (= H21) is a transfer function, and H_{22} is the mid-span driving-point FRF, hence magnitude plot is:



(ii)(iii) Frequency resolution $= f_3/N$



Modal bandwidth, $\Delta f_1 = n \frac{f_3}{N} \Rightarrow Q_1 = \frac{f_1}{\Delta f}$

Examiner's comments:

Q1 – Drum with coupled circular membranes
23 attempts average 13.7/20
Popular and generally done well, although a surprising number got lost in the algebra of part (a). Full marks were awarded for effective application of the method of separation of variables and demonstrating the physical understanding required in parts (b) and (c).

Q2 – Interlacing for string on a spring
24 attempts average 13.0/20
Popular. Parts (a) and (b) were generally done well, although many introduced sign errors in formulating the equilibrium boundary condition and manipulating the resulting equations. Part (d) proved particularly challenging, with only a few producing sensible sketches, and many overlooked the rigid-body mode of the added mass in part (e)

Q3 – Damping in a beam – complex modulus
12 attempts average 14.0/20
Less popular but those that did it went pretty well. The start was easy enough, deriving an expression for natural frequency. The next part was a "show that" which was algebraically messy but most candidates made a good start and it was clear that they were going to get the right answer but for algebraic slipups. Disappointing was the application to a particular beam. Many didn't get their head around the question but I think the wording is clear.

Q4 – Instrumentation and modal circles
25 attempts average 15.1/20
Well done, perhaps because this type of question that turns up often. Most regurgitated the usual stuff on the pros and cons of hammers, shakers, accelerometers, lasers but didn't say anything about the actual case study (a painting in the Fitz). Those that did got substantially better marks in part (a). Modal circles and transfer functions in part (b) was well done, but noting that we're plotting acceleration and not velocity or displacement caught many out.