EGT3
ENGINEERING TRIPOS PART IIB

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Tuesday 27 April 2021 1.30 to 3.10

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**Module 4C6**

**ADVANCED LINEAR VIBRATIONS**

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number* ***not*** *your name on the cover sheet and at the top of each answer sheet..*

**STATIONERY REQUIREMENTS**
Write on single-sided paper.

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (9 pages)
You are allowed access to the electronic version of the Engineering Data Books.

**10 minutes reading time is allowed for this paper at the start of the exam.**

**The time taken for scanning/uploading answers is 15 minutes.**

**Your script is to be uploaded as a single consolidated pdf containing all answers.**

1 A double headed drum consists of two circular membranes that are mounted at each end of a cylinder and coupled via the enclosed air. The out-of-plane displacement (in polar coordinates $r$ and $\theta$) of the top membrane is $u_1(r,\theta)$ and the out-of-plane displacement of the lower membrane is $u_2(r,\theta)$. To a first approximation the coupled differential equations of motion of the membranes are

$$T\left(\frac{\partial^2 u_1}{\partial r^2}+\frac{1}{r}\frac{\partial u_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2}\right)=m\frac{\partial^2 u_1}{\partial t^2}+\lambda(u_1-u_2)\,,$$

$$T\left(\frac{\partial^2 u_2}{\partial r^2}+\frac{1}{r}\frac{\partial u_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_2}{\partial \theta^2}\right)=m\frac{\partial^2 u_2}{\partial t^2}+\lambda(u_2-u_1)\,,$$

where $T$ is the membrane tension per unit length, $m$ is the mass per unit area, and $\lambda$ is a constant representing the compressibility of the enclosed air. Each membrane has radius $a$ and is restrained from motion on the boundary $r=a$. The natural frequencies and mode shapes of the drum are sought by assuming a separable solution of the form

$$u_1(r,\theta)=A_1 f(r)g(\theta)e^{i\omega t}, \qquad u_2(r,\theta)=A_2 f(r)g(\theta)e^{i\omega t}\,,$$

where $A_1$ and $A_2$ are constants.

(a) Show that a solution of this type is only possible if either $A_1=A_2$ or $A_1=-A_2$. Also show that the functions $g(\theta)$ and $f(r)$ must satisfy the equations

$$g''+n^2 g=0\,,$$

$$r^2 f''+rf'+(k^2r^2-n^2)f=0\,,$$

where $n$ is an integer, and give the *two* possible expressions for $k$ in terms of $T$, $m$, $\omega$, and $\lambda$. [40%]

(b) Given that the equation that governs $f(r)$ is Bessel's differential equation, show, with reference to the data sheet, that two of the natural frequencies are given by

$$\omega_1^2=\frac{T}{m}\left(\frac{2.404}{a}\right)^2, \qquad \omega_2^2=\frac{T}{m}\left(\frac{2.404}{a}\right)^2+\frac{2\lambda}{m}\,,$$

and give the corresponding mode shapes. Explain physically the two natural frequencies that are obtained when $T\to 0$. [40%]

(c) Find the lowest natural frequency of the drum for the case in which the lower membrane is restrained from moving. [20%]

2 A string of length $L$ with tension $T$ and mass per unit length $m$ is restrained from motion at each end. The coordinate $x$ is measured along the string from the left-hand end, and a spring of stiffness $\lambda$ is attached at the location $x = \alpha L$, as shown in Fig. 2. The lateral displacement of the string is $u(x,t)$.

(a) Show that a solution of the form

$$u(x,t) = \begin{cases} A\sin kx \sin \omega t & x < \alpha L \\ B\sin k(x-L)\sin \omega t & x \geq \alpha L \end{cases}$$

satistifies the differential equation of motion and the boundary condition at each end of the string providing $k = \omega\sqrt{m/T}$ . [10%]

(b) The required conditions at $x = \alpha L$ are continuity of displacement, and equilibrium between the force in the spring and the sum of the resolved component of tension in the string on each side of the spring. Express these two conditions mathematically, and show that they will be satisfied providing that

$$kT \sin kL = -\lambda \sin k\alpha L \sin k(1-\alpha)L .$$ [40%]

(c) Show that the equation found in part (b) yields the correct solutions for the natural frequencies of the system for each of the two limiting cases $\lambda = 0$ and $\lambda = \infty$, given that the natural frequencies of a string of length $L$ are $\omega_n = (n\pi / L)\sqrt{T/m}$ for integer values of $n$. [15%]

(d) Given that the natural frequencies obtained for $\lambda = \infty$ interlace with those obtained for $\lambda = 0$, show by graphical means that the natural frequencies obtained for a general value of $\lambda$ must interlace with those obtained for $\lambda = 0$. Explain how this result could have been predicted by the interlacing theorem without the need for any mathematical analysis. [20%]

(e) How would the predictions of the interlacing theorem change if the spring in Fig. 2 were replaced by a mass attached to the string? [15%]

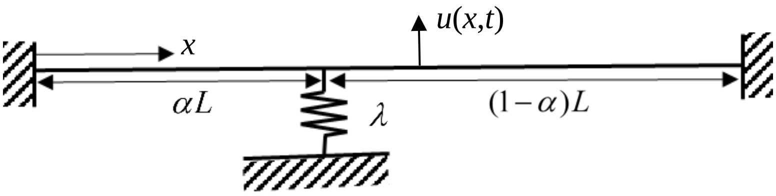


Fig. 2

3 The differential equation of motion that governs the lateral displacement $u(x,t)$ of a beam that has both bending and shear deformation has the form

$$EI\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} - \frac{mEI}{\kappa AG}\frac{\partial^4 u}{\partial x^2 \partial t^2} = 0,$$

where $EI$ is the flexural rigidity, $m$ is the mass per unit length, $A$ is the cross-sectional area, $G$ is the shear modulus, and $\kappa$ is a constant that depends on the shape of the cross-section. It is found that the vibration of the beam in a particular natural mode has the form

$$u(x,t) = \sin px \sin \omega t,$$

where $p$ is the wavenumber of the spatial deformation.

(a) Derive an expression for the natural frequency of the mode in terms of the wavenumber $p$. [10%]

(b) To incorporate the effect of damping the moduli of the material are replaced by complex values, so that $E \to E_0(1+i\eta_E)$ and $G \to G_0(1+i\eta_G)$. For small values of the loss factors $\eta_E$ and $\eta_G$, show that the loss factor of the mode of vibration of the beam is given approximately by

$$\eta = \frac{m\,\eta_E + \alpha p^2 \eta_G}{m + \alpha p^2},$$

where $\alpha = mE_0 I/(\kappa A G_0)$. [40%]

(c) A beam of square cross-section with side length 20 mm has $\kappa = 5/6$. If the beam is made of isotropic material with Poisson ratio $\nu = 0.3$ calculate the wavelength of the mode for which the modal loss factor arising from the shear modulus is equal to that arising from the Young's modulus for the case $\eta_G = \eta_E$. [25%]

(d) For $\eta_E \neq \eta_G$ show that the modal loss factor either increases constantly with increasing $p$ or decreases constantly with increasing $p$. [25%]

4 (a) The Fitzwilliam Museum is sending a valuable painting (in its frame) on loan for an exhibition in the USA. They are worried about the effect of vibration on the painting during transport. You have been asked to do vibration measurements on the painting for the purposes of modal analysis. For this application:

(i) Describe the advantages and disadvantages of using an instrumented hammer versus an electromagnetic shaker as an excitation device. [20%]

(ii) Describe the advantages and disadvantages of using an accelerometer versus a laser-Doppler vibrometer for response measurement. [20%]

(iii) Describe two simple checks to test that an accelerometer-based measurement system is functioning correctly. [10%]

(b) An impulse hammer and accelerometer are used to investigate the response of a simple footbridge, which behaves as a simply-supported beam, as sketched in Fig. 4. The accelerometer is attached at the mid-span location (Point 2), and impulses are applied at the mid-span and at a quarter-point (Point 1).

(i) On a decibel vertical scale, sketch the expected form of the magnitude of the frequency-response functions $H_{12}$ and $H_{22}$ over a frequency range that includes the first three natural frequencies. Label salient features. [20%]

(ii) Sketch the corresponding Nyquist plot for the frequency-response function $H_{12}$, labelling salient features. [15%]

(iii) If $N$ data points are acquired per channel at a sampling frequency of $f_S$ describe how you would use such plots to obtain a best-estimate of the Q-factor of the fundamental mode of the footbridge. [15%]

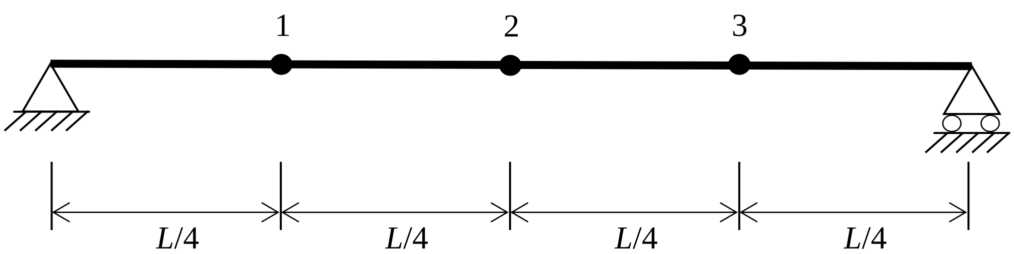


Fig. 4

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## Part IIB Data sheet

## Module 4C6 Advanced linear vibration

### VIBRATION MODES AND RESPONSE

**Discrete systems**

**1.** The forced vibration of an $N$-degree-of-freedom system with mass matrix $M$ and stiffness matrix $K$ (both symmetric and positive definite) is

$$M\,\underline{\ddot{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

**2. Kinetic energy**

$$T = \frac{1}{2}\underline{\dot{y}}^t M \underline{\dot{y}}$$

**Potential energy**

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

**3.** The natural frequencies $\omega_n$ and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} \; .$$

**4. Orthogonality and normalisation**

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

**Continuous systems**

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 4 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 \, dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies $\omega_n$ and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

**5. General response**

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\, \underline{u}^{(j)} = U\underline{q}(t)$$

where $U$ is a matrix whose $N$ columns are the normalised eigenvectors $\underline{u}^{(j)}$ and $q_j$ can be thought of as the "quantity" of the $j$th mode.

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\, u_j(x)$$

where $w(x,t)$ is the displacement and $q_j$ can be thought of as the "quantity" of the $j$th mode.

**6.** Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f}\ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2\, q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

**7. Frequency response function**

For input generalised force $f_j$ at frequency $\omega$ and measured generalised displacement $y_k$ the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force $F$ at frequency $\omega$ applied at point $x$, and displacement $w$ measured at point $y$, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

**8. Pattern of antiresonances**

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

## 9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response $y_k$ is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point $x$, the response at point $y$ is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

## 10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ the measured response $y_k$ is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point $x$, the response at point $y$ is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

## Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, $M$ is the mass matrix and $K$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 4.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to $\omega_k^2$ if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

## GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

### Transverse vibration of a stretched string

Tension $P$, mass per unit length $m$, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

### Torsional vibration of a circular shaft

Shear modulus $G$, density $\rho$, external radius $a$, internal radius $b$ if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$ |

### Axial vibration of a rod or column

Young's modulus $E$, density $\rho$, cross-sectional area $A$, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

### Bending vibration of an Euler beam

Young's modulus $E$, density $\rho$, cross-sectional area $A$, second moment of area of cross-section $I$, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

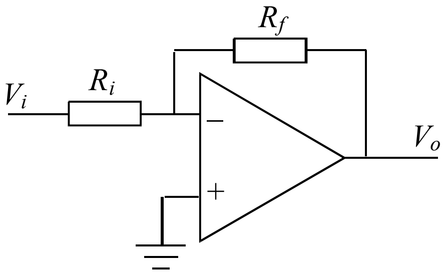
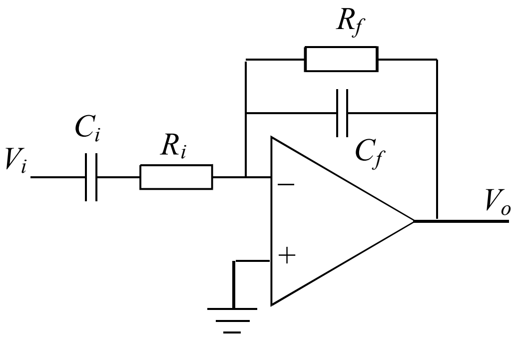
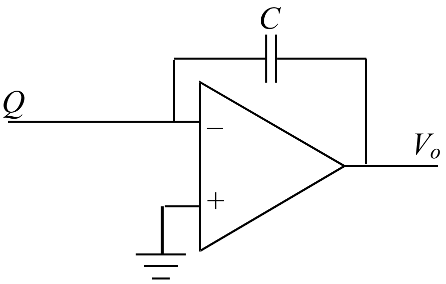
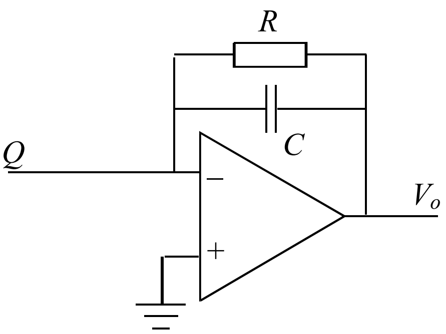
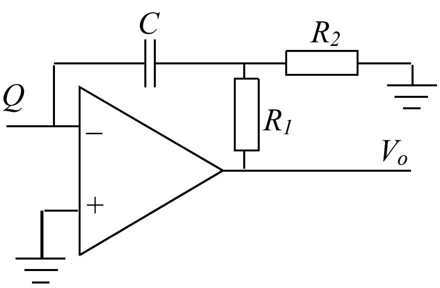
| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$ | $V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$ |

Note that values of $I$ can be found in the Mechanics Data Book.

## VIBRATION MEASUREMENT

### Some useful OpAmp circuits for instrumentation

(Note: $j$ is used instead of $i$ here for $\sqrt{-1}$ for compatibility with the Electrical Data Book.)

| | |
|---|---|
|  | Inverting voltage amplifier<br>$V_o = -\frac{R_f}{R_i} V_i$ |
|  | Inverting voltage amplifier with low-pass and high-pass filter<br>$V_o = -\frac{R_f}{R_i} \frac{V_i}{(1 + \frac{1}{j\omega R_i C_i})(1 + j\omega R_f C_f)}$ |
|  | Inverting charge amplifier<br>$V_o = -\frac{Q}{C}$ |
|  | Inverting charge amplifier with high-pass filter<br>$V_o = -\frac{Q}{C} \frac{1}{1 + \frac{1}{j\omega RC}}$ |
|  | Inverting charge amplifier with additional gain<br>$V_o = -\frac{Q}{C} \frac{R_1 + R_2}{R_2}$ |

## Some devices for vibration excitation and measurement

*Moving coil electro-magnetic shaker*

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

*Piezo stack actuator*

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1μm, u
nloaded resonance 400kHz

*Impulse hammer*

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

*Piezo accelerometer*

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

*MEMS accelerometer*

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

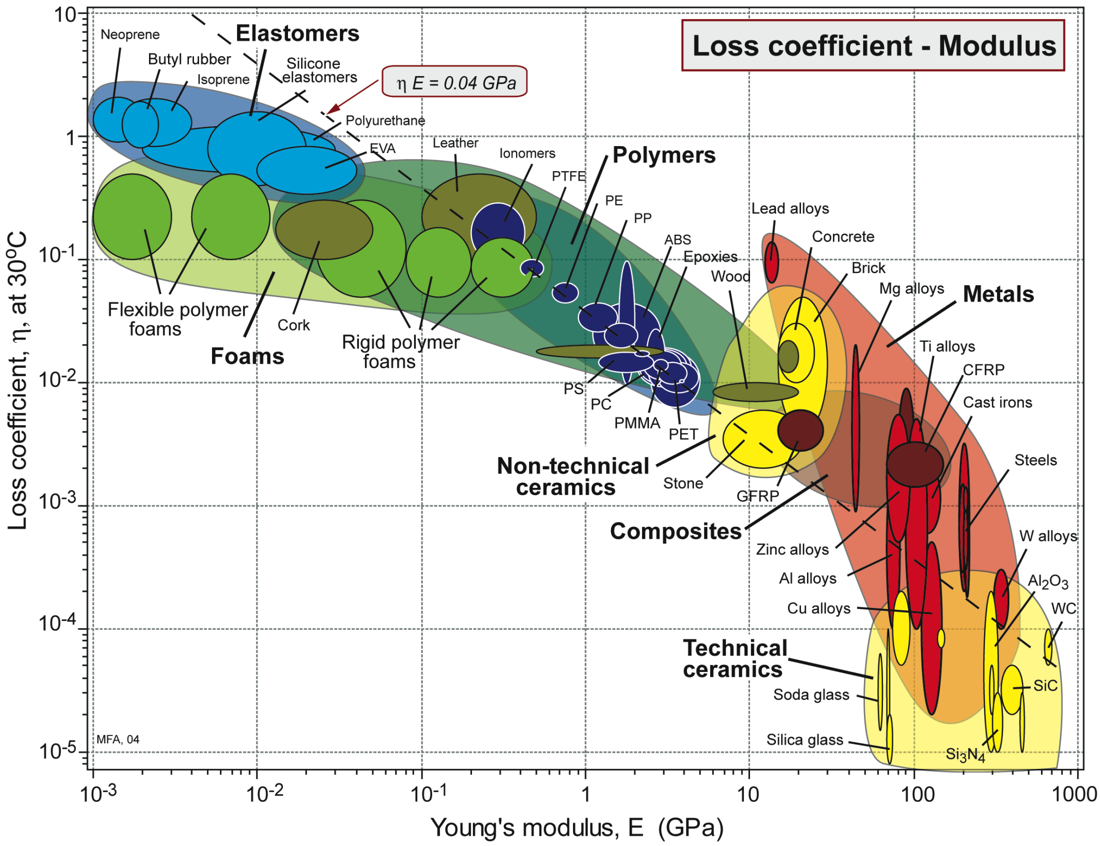
*Laser Doppler Vibrometer*

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

## VIBRATION DAMPING

### Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \to E(1+i\eta)$. Typical values of $E$ and $\eta$ for engineering materials are shown below:




For a complex natural frequency $\omega$:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

## Free and constrained layers

For a 2-layer beam: if layer $j$ has Young's modulus $E_j$, second moment of area $I_j$ and thickness $h_j$, the effective bending rigidity $EI$ is given by:

$$EI = E_1 I_1 \left[ 1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh} \right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1} .$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12} \left[ \frac{h_{31} - d}{1+g} \right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[ \frac{E_2 h_2}{2} (h_{21} - d) + E_3 h_3 (h_{31} - d) \right] \left[ \frac{h_{31} - d}{1+g} \right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

$G_2$ is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

## Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t \, C \, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and $C$ is the (symmetric) dissipation matrix.

If the system has mass matrix $M$ and stiffness matrix $K$, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1} K & -M^{-1} C \end{bmatrix}$$

where 0 is the zero matrix and $I$ is the unit matrix.

## THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume $V$ with a neck of effective length $L$ and cross-sectional area $S$ has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where $c$ is the speed of sound in air.

The end correction for an unflanged circular neck of radius $a$ is $0.6a$.

The end correction for a flanged circular neck of radius $a$ is $0.85a$.

## VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension $T$ and mass per unit area $m$ undergoes small transverse free vibration with displacement $w$, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates $x, y$ or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates $r, \theta$.

For a circular membrane of radius $a$ the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where $J_n$ is the Bessel function of order $n$ and $k$ is determined by the condition that $J_n(ka) = 0$. The first few zeros of $J_n$'s are as follows:

| | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---|---|---|---|---|
| $ka =$ | 2.404 | 3.832 | 5.135 | 6.379 |
| $ka =$ | 5.520 | 7.016 | 8.417 | 9.760 |
| $ka =$ | 8.654 | 10.173 | | |

For a given $k$ the corresponding natural frequency $\omega$ satisfies

$$k = \omega\sqrt{m/T}\ .$$