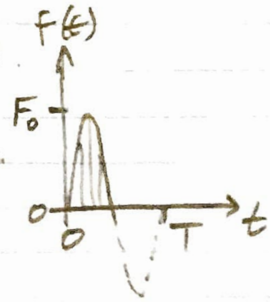
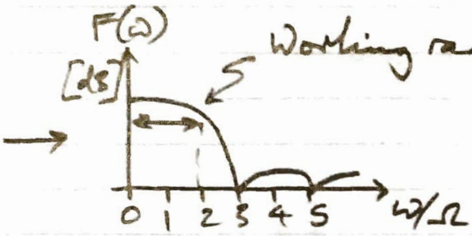
4C6: Advanced Linear Vibration 2022

1 a).





Working range given by $\frac{\omega}{\Omega} \simeq 2$, i.e. $\omega_{max} = 2\Omega$

$\Omega = \sqrt{\frac{k}{m}}$

k = hammer tip stiffness
m = " mass

$f(t) = F_0 \sin \Omega t$, $T = 2b = \frac{2\pi}{\Omega}$

$\omega_{max} = 2\Omega$, $f_{max} = \frac{\Omega}{\pi}$ $\therefore$ $b = \frac{\pi}{\Omega} = \frac{1}{f_{max}}$

So, to measure up to the third mode, require $b = \frac{1}{f_{max}} = \frac{1}{5.6}$

$= 0.179\,s$

$= 179\,ms$ [3]

b) For $m = 3\,kg$ and $\Omega = \pi f_{max} = \sqrt{\frac{k}{m}}$ $\Rightarrow$ $k = (\pi f_{max})^2 m$

$= (5.6\pi)^2 \times 3 = 929\,N/m$

[2]

c). Sensitivity $= 0.2\,mV/N$

For hammer tip compression $x = X \sin \Omega t$, tip velocity $\dot{x} = \Omega X \cos \Omega t$

$\Rightarrow V = \dot{x}(0) = \Omega X$

$\left.\begin{array}{l} \therefore \text{ peak force } F_0 = kX = \frac{kV}{\Omega} = m\Omega V \\ \text{Hammer falls } h \quad \therefore V = \sqrt{2gh} \end{array}\right\} \Rightarrow F_0 = m\Omega\sqrt{2gh}$

$= 3 \times \pi \times 5.6 \sqrt{2 \times 9.81 \times 1.5}$

$= 286\,N$

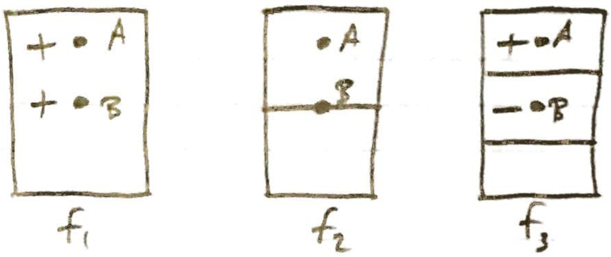
Hence expect output signal to peak at $0.2 \times 286 = 57\,mV$ [3]

d). Nyquist $\Rightarrow f_{Nyq} = 2 f_{max} = 2 \times 5.6 = 11.2\,Hz$ as minimum sampling frequency.

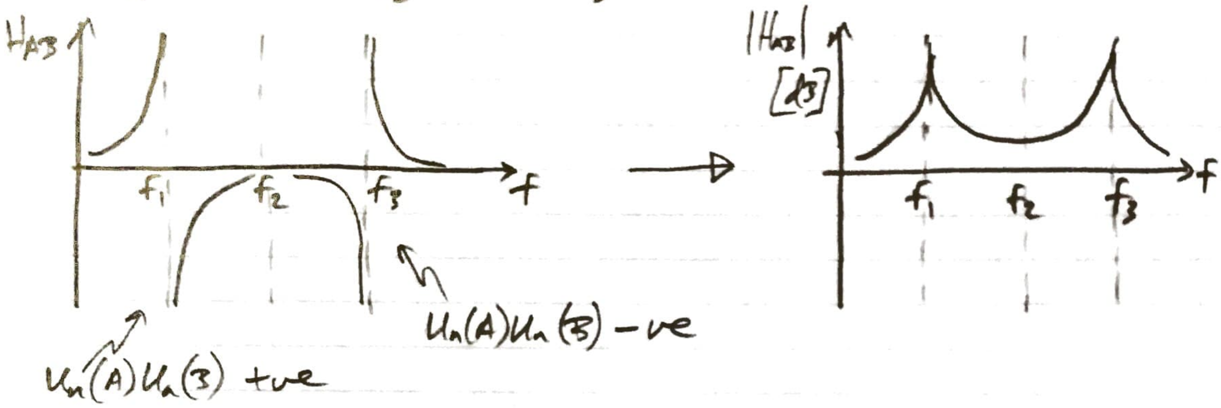
$\therefore$ $f_s = 30\,Hz$ provides ample margin to avoid ailiasing without the need for a low-pass filter. [1]

① cont.

e).



B lies on nodal line of Mode 2 so this does not feature in H_{AB}



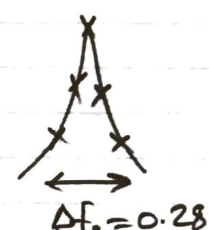
[3]

f).

$$Q_3 = 20, \; f_3 = 5.6 \text{ Hz} \Rightarrow \Delta f_3 = \frac{f_3}{Q_3} = \frac{5.6}{20} = 0.28$$

To confirm the Q-factor, we require adequate spectral resolution to define the modal peak.

Consider a minimum of 5 points around the peak:

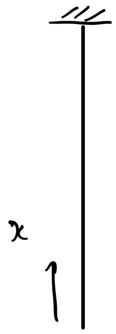


This would require a frequency resolution of $\frac{0.28}{4} = 0.07$ Hz and hence a recording length of $T = \frac{1}{0.07} = 14$ s. This is too long: the floor response will have decayed before this. The impulse method is therefore unable to excite sufficiently long transients and we must consider an alternative method based on a shaker or making use of random ambient vibration.

[3]

4C6 Cribs 2022

2 a)



$$T(x) = Mgx$$

assuming harmonic motion

$$\Rightarrow \quad \frac{d}{dx}\left(Mgx\frac{du}{dx}\right) + M\omega^2 u = 0$$

$$\Rightarrow \quad Mg\frac{du}{dx} + Mgx\frac{d^2u}{dx^2} + M\omega^2 u = 0 \qquad \text{— (1)}$$

Put $z = x^{1/2}$ $\Rightarrow$ $\dfrac{du}{dx} = \dfrac{du}{dz}\dfrac{dz}{dx} = \dfrac{du}{dz}\,\tfrac{1}{2}x^{-1/2} = \dfrac{du}{dz}\,\tfrac{1}{2}\left(\tfrac{1}{z}\right)$

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dz^2}\,\tfrac{1}{4}x^{-1} + \frac{du}{dz}\left(-\tfrac{1}{4}\right)x^{-3/2}$$

$$= \frac{d^2u}{dz^2}\,\tfrac{1}{4}\left(\tfrac{1}{z}\right)^2 - \tfrac{1}{4}\frac{du}{dz}\left(\tfrac{1}{z}\right)^3$$

$$\Rightarrow \quad Mg\frac{du}{dx} + Mgx\frac{d^2u}{dx^2} = Mg\,\tfrac{1}{2}\frac{du}{dz}\left(\tfrac{1}{z}\right) + Mg\frac{d^2u}{dz^2}\,\tfrac{1}{4} - \tfrac{1}{4}\frac{du}{dz}\,Mg\left(\tfrac{1}{z}\right)$$

$$= \tfrac{1}{4}Mg\left(\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz}\right)$$

Substitute in (1) $\Rightarrow$ $\underline{\dfrac{d^2u}{dz^2} + \dfrac{1}{z}\dfrac{du}{dz} + \dfrac{4\omega^2}{g}u = 0}$ [3]

Solutions $\quad J_0(2\omega z/\sqrt{g})\,,\ Y_0(2\omega z/\sqrt{g})$

$\quad\quad\quad\quad\quad\ \ \| \quad\quad\quad\quad\quad\quad \|$

$\quad\quad\quad\quad J_0(2\omega\sqrt{x/g})\,,\ Y_0(2\omega\sqrt{x/g})$

Y_0 is ∞ when $x = 0$, and so cannot apply. The boundary condition of zero force at $x = 0$ is automatically satisfied since $T(0) = 0$.

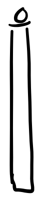
Applying the upper boundary condition: $J_0(2\omega\sqrt{L/g}) = 0$

⇒ From the data sheet $2\omega\sqrt{L/g} = 2.404,\ 5.520,\ 8.654\ldots$

$$\Rightarrow \quad \underline{\omega_1 = 1.202\sqrt{g/L}}$$

[3]

For a rod:

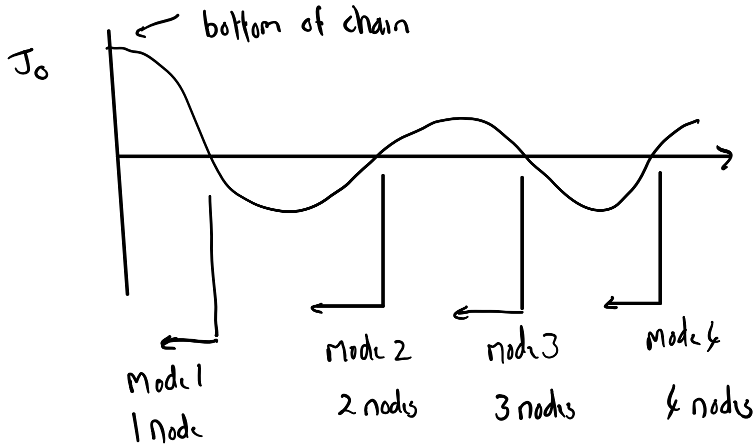


$$\tfrac{1}{3}mL^3\ddot{\theta} + \tfrac{1}{2}mL^2 g\,\theta = 0$$

$$\Rightarrow \quad \omega_n = \sqrt{3/2}\,\sqrt{g/L} = \underline{1.226\sqrt{g/L}}\ \text{, close}$$

[40%]

b)



[20%]

[3]

c) In this case $u = A\,J_0(2\omega\sqrt{x/g}) + B\,Y_0(2\omega\sqrt{x/g})$

Boundary conditions: $A\,J_0(2\omega\sqrt{L/g}) + B\,Y_0(2\omega\sqrt{L/g}) = 0$

$$A\,J_0(2\omega\sqrt{(L-a)/g}) + B\,Y_0(2\omega\sqrt{(L-a)/g}) = 0$$

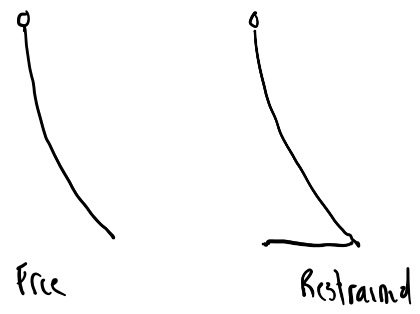
$$\Rightarrow \quad \underline{J_0(2\omega\sqrt{L/g})\,Y_0(2\omega\sqrt{(L-a)/g}) = J_0(2\omega\sqrt{(L-a)/g})\,Y_0(2\omega\sqrt{L/g})}$$

[4]

[25%]

d) This is tricky, $Y_0 \to \infty$ at $x = 0$ and $B \to 0$

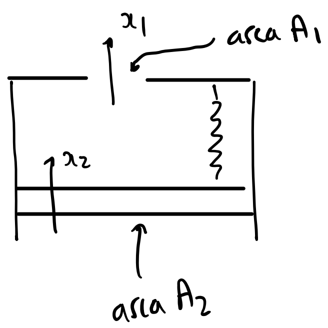
Together they cause a discontinuity at the base of the chain:



[2]

Wn's of the chain are unchanged. [15%]

3. a)



$\Delta V = x_1 A_1 - x_2 A_2$

$\rho = \frac{\text{Air mass}}{V} \Rightarrow \Delta\rho = -\frac{\text{Air mass}}{V^2}\,\Delta V$

$\Rightarrow \Delta\rho = -\rho\,(x_1A_1 - x_2A_2)/V$

$\Rightarrow \Delta p = -\rho c^2 (x_1A_1 - x_2A_2)/V$ [2]

Equation of motion for air mass $M_a \ddot{x}_1 = \Delta p\, A_1$

For bottom plate $M\ddot{x}_2 = -\Delta p\, A_2 - k x_2$

$$\Rightarrow \underline{\begin{pmatrix} M_a & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} \rho c^2 A_1^2/V & -\rho c^2 A_1 A_2/V \\ -\rho c^2 A_1 A_2/V & \rho c^2 A_2^2/V + k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$ [2]

For $x_2 = 0$ $M_a \ddot{x}_1 + \rho c^2 A_1^2/V = 0$

$\downarrow$

$\rho A_1 L_{eff}$; $L_{eff} = 2 \times 0.85 \times a_1$ (two "Flanged" holes)

$\ddot{x}_1 + (c^2 A_1 / V L_{eff})\, x_1 = 0$

$\Rightarrow \underline{\omega_n = c\sqrt{\frac{A_1}{V L_{eff}}}}$ agrees with p9 of data sheet with $A_1 = S$ [35%] [2]

b) $a_1 = 1.5 \times 10^{-2}$, $a_2 = 0.1$ $\Big\}$ $A_1 = \pi a_1^2 = 7.068 \times 10^{-4}$, $A_2 = \pi a_2^2 = 0.03146$

$M_a = \rho \times A_1 \times 2 \times 0.85 \times a_1 \simeq 2.18 \times 10^{-5}$

$M = 0.2$ $k = 5 \times 10^4$, $c = 340$

$$\Rightarrow \begin{pmatrix} 2.18\times10^{-5} & 0 \\ 0 & 0.2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 6.987 & -310.99 \\ -310.99 & 6.384\times10^{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|\underbrace{-\omega^2 M + k}_{\text{matrices above}}| = 0$$

$$(-2.18\times10^{-5}\omega^2 + 6.987)(-0.2\omega^2 + 6.384\times10^{4}) - 310.99^2 = 0$$

$$4.36\times10^{-6}\omega^4 - 2.789\omega^2 + 3.493\times10^{5} = 0$$

$$\omega^2 = \left(2.789 \pm [2.789^2 - 4\times3.493\times0.436]^{1/2}\right)/(2\times4.36\times10^{-6})$$

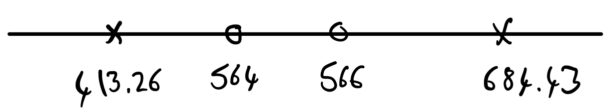
$$\omega^2 = (2.789 \pm 1.2987)\times1.146\times10^{5}$$

$$\Rightarrow \quad \omega = \begin{cases} \underline{413.26} \text{ rad/s} \\ \underline{684.43} \text{ rad/s} \end{cases}$$

[4]

c) For $x_2 = 0$ $\quad \omega_n^2 = 6.987/2.18\times10^{-5} \Rightarrow \underline{\omega_n = 566.13}$

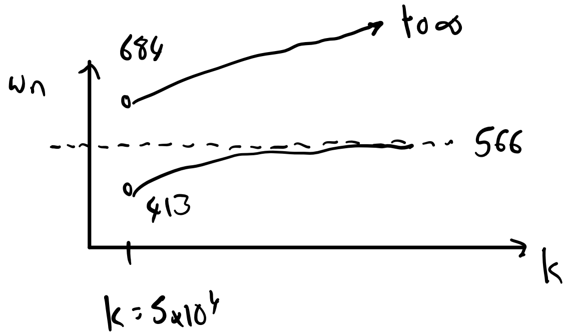
$x_1 = 0$ $\quad \omega_n^2 = 6.384\times10^{4}/0.2 \Rightarrow \underline{\omega_n = 564.97}$



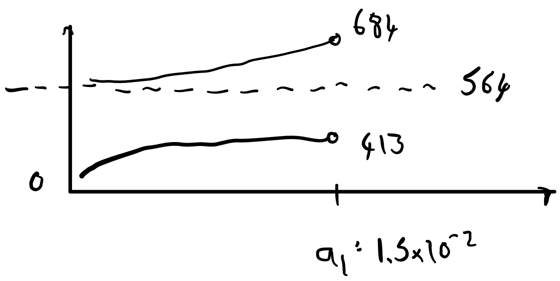
Constrained frequencies <u>interlace</u> with unconstrained results

[3]

d)



Against k



Against a_1

[2]

4 a) The boundary conditions are $\theta(0) = \theta(L) = 0$

For $\sin kL = 0 \Rightarrow k = \frac{n\pi}{L}$ [1]

For mode n, $\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 = \left[-\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\right]^2 = \left(\frac{n\pi}{L}\right)^4 \sin^2\left(\frac{n\pi x}{L}\right)$

$$\left(\frac{\partial\theta}{\partial x}\right)^2 = \left[\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right]^2 = \left(\frac{n\pi}{L}\right)^2 \cos^2\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad V = \tfrac{1}{2} E\Gamma \left(\frac{n\pi}{L}\right)^4 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx + \tfrac{1}{2} GJ \left(\frac{n\pi}{L}\right)^2 \cos^2\left(\frac{n\pi x}{L}\right) dx$$

$$V = \tfrac{1}{4} L \left\{ E\Gamma \left(\frac{n\pi}{L}\right)^4 + GJ \left(\frac{n\pi}{L}\right)^2 \right\}$$

$$\hat{T} = \tfrac{1}{2} \rho J \int_0^L \theta^2 dx = \tfrac{1}{4} L \rho J$$

Rayleigh quotient $\underline{\omega_n^2 = V/\hat{T} = \left[E\Gamma \left(\frac{n\pi}{L}\right)^4 + GJ \left(\frac{n\pi}{L}\right)^2\right] / (\rho J)}$ [4]

b) $E \rightarrow E(1 + i\eta_E)$ $\quad G \rightarrow G(1 + i\eta_G)$

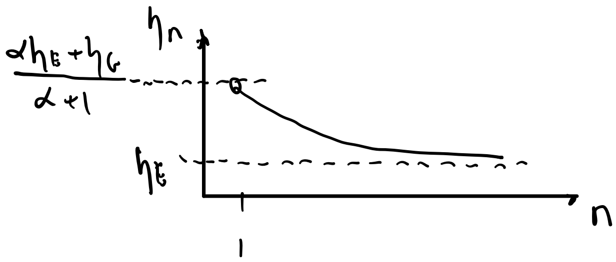
$$\eta_n = \mathrm{Im}(\omega_n^2) / \mathrm{Re}(\omega_n^2)$$

$$\eta_n = \left[E\Gamma \eta_E \left(\frac{n\pi}{L}\right)^4 + GJ \eta_G \left(\frac{n\pi}{L}\right)^2\right] / \left[E\Gamma \left(\frac{n\pi}{L}\right)^4 + GJ \left(\frac{n\pi}{L}\right)^2\right]$$

$$\Rightarrow \quad \underline{\eta_n = \left(\frac{\alpha n^2 \eta_E + \eta_G}{\alpha n^2 + 1}\right)} \qquad \alpha = \left(\frac{E\Gamma}{GJ}\right)\left(\frac{\pi}{L}\right)^2$$ [4]

Now $\eta_G > \eta_E$, as n increases result has a greater weighting from η_E

$\Rightarrow$ η_n reduces with increasing n, to an asymptote of η_E [1]



c) $\omega_n^2 \to \omega_n^2(1+i\eta_n) \Rightarrow \omega_n \to \omega_n(1+\frac{1}{2}\eta_n i)$

$$e^{i\omega_n t} \to \underbrace{e^{-\frac{1}{2}\eta_n\omega_n t}}_{\text{decay rate}} e^{i\omega_n t}$$

[1]

$$\eta_n\omega_n = \underbrace{\left(\frac{\alpha n^2\eta_E+\eta_G}{\alpha n^2+1}\right)}_{\to\ \eta_E} \times \underbrace{\left[(\alpha n^2+1)\left(\frac{GJ}{\rho J}\right)\left(\frac{n\pi}{L}\right)^2\right]^{1/2}}_{\omega_n\ \to\ n^2}$$

For $n \to \infty$

Decay rate very rapid for large n, due to constant η_n and increasing ω_n [2]

d) For $\mu = 0$ then $\alpha = 0$ $\Rightarrow$ $\eta_n = \eta_G$, independent of n

$$\omega_n = \sqrt{\frac{GJ}{\rho J}}\left(\frac{n\pi}{L}\right)$$

$$\omega_n\eta_n \propto n \times \eta_G$$

Decay rate still increases with n, but not so rapidly

[2]

ENGINEERING TRIPOS PART IIB 2022
ASSESSOR'S REPORT, MODULE 4C6

The course was taken by 23 candidates, of whom 22 took Part IIA and 1 withdrew.

No scaling of the raw marks was required this year.

Q1
This was generally done well, with the majority of answers reflecting a good or better understanding of the principles behind impulse testing. Part (c) required the most thought and was the least well done; part (f) also highlighted some confusion over the relationship between recording duration and frequency resolution.

Q2
This question was the least popular, being attempted by just 8 candidates. Some did poorly, finding the calculus and algebra of part (a) and the Bessel functions of (b) and (c) challenging (there was some confused interpretation of the solutions for a membrane). Only a few made progress with (d) and (e).

Q3
This was the most popular and well done question. A good, balanced question, with the majority of candidates gaining marks in all parts.

Q4
This was also generally done well, reflecting a good understanding of Rayleigh's principle and the correspondence principle. Most lost marks in part (c) and (d), in simplifying the expression for large *n*, and in overlooking the dependency of the decay rate on both loss factor *and* frequency.

J P Talbot