Version JPT/4

EGT3
ENGINEERING TRIPOS PART IIB

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Tuesday 26 April 2022 2 to 3.40

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**Module 4C6**

**ADVANCED LINEAR VIBRATIONS**

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided script paper

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Attachment: 4C6 Advanced Linear Vibration data sheet (9 pages)
Engineering Data Book

**10 minutes reading time is allowed for this paper at the start of the exam.**

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.**

**You may not remove any stationary from the Examination Room.**

1 An impulse test is to be conducted on a floor of a building, for which the first three vibration modes are expected to be as shown in the floor plans (seen from above) of Fig. 1. The impulse hammer consists of a falling mass of 3 kg that slides smoothly down a guide rail to strike the floor before rebounding and being restrained by a catch mechanism.

(a) Using sketches to illustrate your answer, estimate the duration of the impulse required to excite the first three modes whilst minimising any excitation of higher modes. [20%]

(b) Determine the corresponding stiffness of the hammer tip. [10%]

(c) The force transducer has a sensitivity of 0.2 mV/N. If the hammer is allowed to fall from a height of 1.5 m, estimate the peak force of the impulse and hence determine the expected peak output signal from the hammer. [15%]

(d) Explain why 30 Hz would be a suitable sampling frequency. [5%]

(e) Sketch and explain the form of the magnitude of the transfer function that might be measured by attaching an accelerometer to the floor at point A (shown in Fig. 1) and applying an impulse at point B. [20%]

(f) The Q-factor of Mode 3 is thought to be approximately 20. Using a sampling frequency of 30 Hz, estimate the recording time required to measure the transfer function in part (e) and thereby confirm the Q-factor. Explain why this might pose a problem and suggest an alternative method to avoid this. [30%]

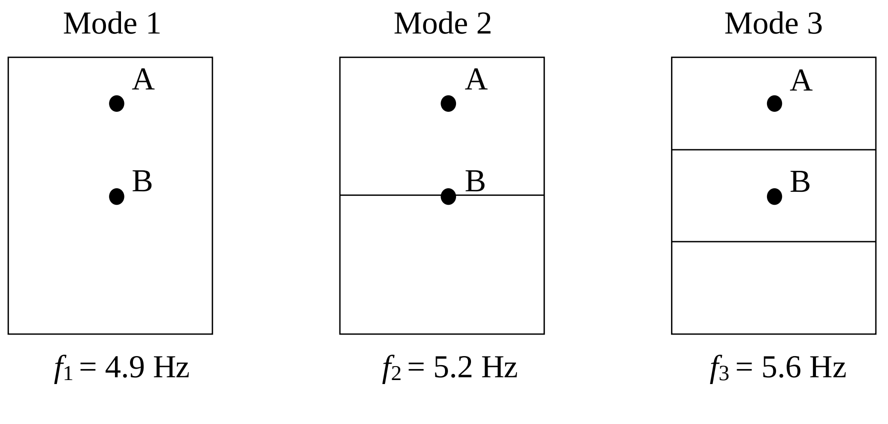


Figure 1

2 The differential equation governing the transverse displacement $u$ of an undamped string with variable tension has the form

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial u}{\partial x}\right]-m\frac{\partial^2 u}{\partial t^2}=0,$$

where the coordinate $x$ is measured along the string, $T(x)$ is the tension at $x$, and $m$ is the mass per unit length. This equation is to be applied to a hanging chain of length $L$, the top end of which is attached to the ceiling and the lower end of which is free and suspended above the floor; the coordinate $x$ is measured upwards, with $x = 0$ at the lower end of the chain.

(a) By assuming harmonic motion of frequency $\omega$ and making the substitution $z=\sqrt{x}$ show that the differential equation that governs the motion of the chain can be written in the form

$$\frac{\partial^2 u}{\partial z^2}+\frac{1}{z}\frac{\partial u}{\partial z}+\frac{4\omega^2}{g}u=0\,.$$

[20%]

(b) This is Bessel's differential equation, as given in the data sheet for the case of $n = 0$.

(i) Write down the two possible solutions to the equation and explain why one of the solutions cannot contribute to the response for the stated boundary conditions. [10%]

(ii) By applying the boundary condition at the top of the chain show that the first natural frequency is given by $\omega_1=1.202\sqrt{g/L}$, and compare this result with the natural frequency of a hinged rigid rod having the same mass per unit length as the chain. [10%]

(c) By considering the shape of the Bessel function of order zero, show that the $n$th mode shape of the chain has $n$ nodes, including the node at the ceiling. [20%]

(d) The chain is now restrained from moving at a point which is a distance $a$ below the ceiling. Show that the natural frequencies of the upper part of the chain (between the restraint and the ceiling) are given by the solutions to the equation

$$\mathrm{J}_0(2\omega\sqrt{L/g})\mathrm{Y}_0(2\omega\sqrt{(L-a)/g})=\mathrm{J}_0(2\omega\sqrt{(L-a)/g})\mathrm{Y}_0(2\omega\sqrt{L/g})\,.$$ [25%]

(e) Discuss the situation that would arise were the constraint applied at $a=L$. [15%]

3 A cylindrical vessel of radius $a_2$ and volume $V$ has a hole of radius $a_1$ in the top circular plate, as shown in Figure 2. The bottom circular plate has mass $M$ and is constrained to slide axially within the cylinder, being attached to the top plate by a spring of stiffness $K$. The upwards displacement of the bottom plate is $x_2$, and the air surrounding the top circular hole has a Helmholtz displacement $x_1$.

(a) By making use of the relation $\Delta p = c^2 \Delta\rho$, where $c$ is the speed of sound, show that the equations of motion of the system have the form

$$\begin{pmatrix} M_a & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} \rho c^2 A_1^2 / V & -\rho c^2 A_1 A_2 / V \\ -\rho c^2 A_1 A_2 / V & \rho c^2 A_2^2 / V + K \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $M_a$ is the mass of air attributed to the hole in the top plate, and $A_1$ and $A_2$ are respectively the areas of the top hole and the bottom plate. Show that for $x_2 = 0$ the Helmholtz natural frequency predicted by this equation agrees with the formula on the data sheet. [35%]

(b) The density of air is $\rho = 1.21\ \text{kg/m}^3$ and the speed of sound is $c = 340\ \text{m/s}$. The system has dimensions $a_1 = 1.5$ cm, $a_2 = 10$ cm, $V = 0.01\ \text{m}^3$, and the mass and stiffness associated with the bottom plate are $M = 0.2$ kg, $K = 5 \times 10^4$ N/m. Find the two natural frequencies of the system. [35%]

(c) Find the natural frequencies for each of the two constrained cases: (i) $x_1 = 0$, (ii) $x_2 = 0$. Show that these results are consistent with the interlacing theorem. [20%]

(d) Without making further detailed calculations sketch a graph of the two natural frequencies of the system as a function of $K$, with $K$ ranging from the initial value to a very large value. Sketch a second graph of the natural frequencies plotted as a function of $a_1$, with $a_1$ decreasing from the initial value to zero. [10%]

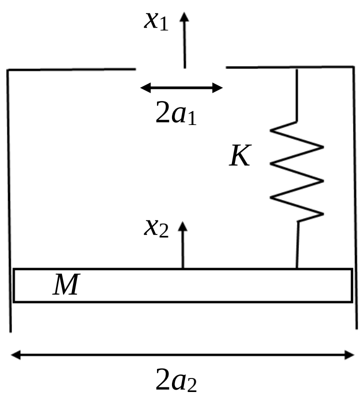


Figure 2

4 When allowance is made for warping of the cross-section, the torsional potential energy of a beam of length $L$ has the form

$$V = (E\Gamma/2)\int_0^L \left(\frac{\partial^2\theta}{\partial x^2}\right)^2 \mathrm{d}x + (GJ/2)\int_0^L \left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x,$$

where the coordinate $x$ is measured along the beam, $E$ and $G$ are the material constants, $\Gamma$ is the warping constant, $J$ is the torsional constant, and $\theta(x,t)$ is the rotation of the cross-section at $x$. The polar moment of inertia of the beam per unit length is $\rho I$.

(a) If the beam is clamped at both ends then the mode shapes have the form $\sin kx$. Specify the values that $k$ can take to satisfy the boundary conditions, and hence use Rayleigh's quotient to find the $n$th natural frequency of the beam. [30%]

(b) The effect of damping is to be modelled by using the correspondence principle, with loss factors $\eta_E$ and $\eta_G$ associated respectively with the material moduli $E$ and $G$ ($\eta_G > \eta_E$). Derive an expression for the loss factor $\eta_n$ of the $n$th mode of vibration. Does the modal loss factor increase or decrease with increasing $n$? [30%]

(c) Show that the decay of vibration in mode $n$ is governed by the factor $\exp(-\eta_n\omega_n t/2)$. Discuss the dependency of the rate of decay on $n$. [25%]

(d) Repeat parts (b) and (c) for the case in which warping does not occur, i.e. $\Gamma = 0$. [15%]

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# Part IIB Data Sheet

# Module 4C6 Advanced Linear Vibration

## 1 Vibration Modes and Response

**Discrete Systems** / **Continuous Systems**

### 1. Equation of motion

**Discrete Systems:** The forced vibration of an $N$-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

**Continuous Systems:** The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 2 for examples.

### 2. Kinetic Energy

**Discrete Systems:**

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

**Continuous Systems:**

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

### 3. Potential Energy

**Discrete Systems:**

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

**Continuous Systems:** See Section 2 for examples.

### 4. Natural frequencies and mode shapes

**Discrete Systems:** The natural frequencies $\omega_n$ and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

**Continuous Systems:** The natural frequencies $\omega_n$ and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 2) and boundary conditions, assuming harmonic time dependence.

### 5. Orthogonality and normalisation

**Discrete Systems:**

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

**Continuous Systems:**

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

## 6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose $N$ columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and $q_j$ can be thought of as the 'quantity' of the $j$th mode.

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_{j} q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and $q_j$ can be thought of as the 'quantity' of the $j$th mode.

## 7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

## 8. Frequency response function

For input generalised force $f_j$ at frequency $\omega$ and measured generalised displacement $y_k$, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force $F$ at frequency $\omega$ applied at point $x_1$, and displacement $y$ measured at point $x_2$, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor $\zeta_n$ is as in the Mechanics Data Book for one-degree-of-freedom systems.

## 9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

**10. Impulse responses**

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response $y_k$ is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point $x_1$, the response at point $x_2$ is

$$g(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} \sin \omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n} e^{-\omega_n \zeta_n t} \sin \omega_n t$$

for $t \geq 0$ (with small damping).

**11. Step response**

For a unit step generalised force $f_j$ applied at $t = 0$, the measured response $y_k$ is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos \omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point $x_1$, the response at point $x_2$ is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \left[1 - \cos \omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \left[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\right]$$

for $t \geq 0$ (with small damping).

## 1.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\widetilde{T}} = \frac{\mathbf{y}^T\mathbf{K}\mathbf{y}}{\mathbf{y}^T\mathbf{M}\mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 2.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to $\omega_k^2$ if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\widetilde{T}}$ is *stationary* near each mode.

# 2 Governing equations for continuous systems

## 2.1 Transverse vibration of a stretched string

Tension $P$, mass per unit length $m$, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$ |

## 2.2 Torsional vibration of a circular shaft

Shear modulus $G$, density $\rho$, external radius $a$, internal radius $b$ if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$ | $V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$ |

## 2.3 Axial vibration of a rod or column

Young's modulus $E$, density $\rho$, cross-sectional area $A$, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$ | $V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$ |

## 2.4 Bending vibration of an Euler beam

Young's modulus $E$, density $\rho$, cross-sectional area $A$, second moment of area of cross-section $I$, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

| Equation of motion | Potential energy | Kinetic energy |
|---|---|---|
| $\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$ | $V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$ | $T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$ |

Note that values of $I$ can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$
\begin{aligned}
\cos\alpha\cosh\alpha + 1 &= 0, \quad \alpha_1 = 1.8751\\
\cos\alpha\cosh\alpha - 1 &= 0, \quad \alpha_1 = 4.7300\\
\tan\alpha - \tanh\alpha &= 0, \quad \alpha_1 = 3.9266
\end{aligned}
$$

## Some devices for vibration excitation and measurement

*Moving coil electro-magnetic shaker*

LDS V101: Peak sine force 10N, internal armature resonance 12kHz. Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, stroke 2.5mm, shaker body mass 0.9kg

LDS V650: Peak sine force 1kN, internal armature resonance 4kHz. Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg

LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz. Frequency range 5 – 1.7kHz, armature suspension stiffness 72kN/m, armature mass 250kg, stroke 50mm, shaker body mass 13000kg

*Piezo stack actuator*

FACE PAC-122C
Size 2×2×3mm, mass 0.1g, peak force 12N, stroke 1µm, u
nloaded resonance 400kHz

*Impulse hammer*

IH101
Head mass 0.1kg, hammer tip stiffness 1500kN/m, force transducer sensitivity 4pC/N, internal resonance 50kHz

*Piezo accelerometer*

B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale range +/-5000g

DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale range +/-2000g

B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, full-scale range +/-2000g

*MEMS accelerometer*

ADKL202E
265mV/g
Full scale range +/- 2g
DC-6kHz

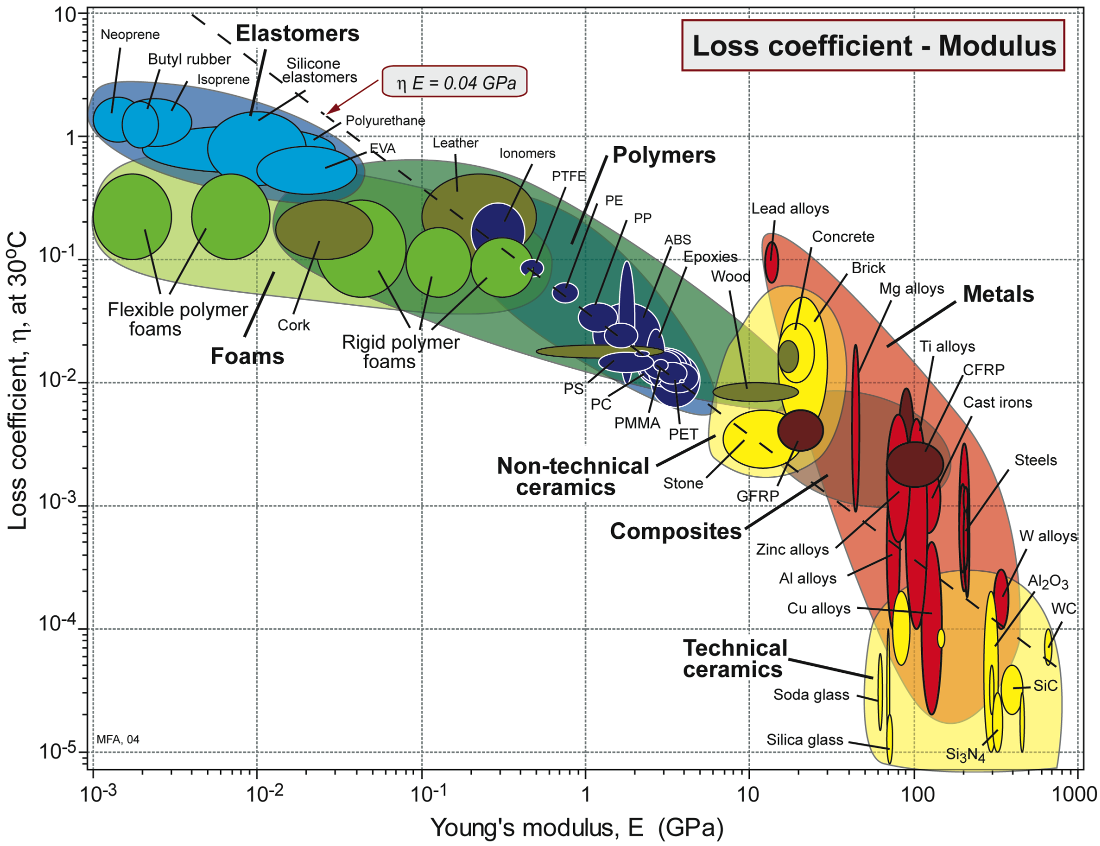
*Laser Doppler Vibrometer*

Polytec PSV-400 Scanning Vibrometer
Velocity ranges 2/10/50/100/1000 [mm/s/V]

**VIBRATION DAMPING**

**Correspondence principle**

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \to E(1+i\eta)$. Typical values of $E$ and $\eta$ for engineering materials are shown below:




For a complex natural frequency $\omega$:

$$\omega \simeq \omega_n\left(1+i\zeta_n\right) \simeq \omega_n\left(1+i\eta_n/2\right) \simeq \omega_n\left(1+i/2Q_n\right)$$

and

$$\omega^2 \simeq \omega_n^2\left(1+i\eta_n\right) \simeq \omega_n^2\left(1+i/Q_n\right)$$

## Free and constrained layers

For a 2-layer beam: if layer $j$ has Young's modulus $E_j$, second moment of area $I_j$ and thickness $h_j$, the effective bending rigidity $EI$ is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh}\right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1} .$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12}\left[\frac{h_{31} - d}{1+g}\right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2}(h_{21} - d) + E_3 h_3 (h_{31} - d)\right]\left[\frac{h_{31} - d}{1+g}\right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

$G_2$ is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

**Viscous damping, the dissipation function and the first-order method**

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2}\underline{\dot{y}}^t\, C\, \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and $C$ is the (symmetric) dissipation matrix.

If the system has mass matrix $M$ and stiffness matrix $K$, free motion is governed by

$$M\,\underline{\ddot{y}} + C\,\underline{\dot{y}} + K\,\underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix}\underline{y} \\ \underline{\dot{y}}\end{bmatrix}$. If $\underline{z} = \underline{u}e^{\lambda t}$ then $\underline{u}, \lambda$ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and $I$ is the unit matrix.

## THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume $V$ with a neck of effective length $L$ and cross-sectional area $S$ has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where $c$ is the speed of sound in air.

The end correction for an unflanged circular neck of radius $a$ is $0.6a$.

The end correction for a flanged circular neck of radius $a$ is $0.85a$.

## VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension $T$ and mass per unit area $m$ undergoes small transverse free vibration with displacement $w$, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates $x, y$ or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates $r, \theta$.

For a circular membrane of radius $a$ the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where $J_n$ is the Bessel function of order $n$ and $k$ is determined by the condition that $J_n(ka) = 0$. The first few zeros of $J_n$'s are as follows:

| | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|---|---|---|---|---|
| $ka =$ | 2.404 | 3.832 | 5.135 | 6.379 |
| $ka =$ | 5.520 | 7.016 | 8.417 | 9.760 |
| $ka =$ | 8.654 | 10.173 | | |

For a given $k$ the corresponding natural frequency $\omega$ satisfies

$$k = \omega\sqrt{m/T}\ .$$