PART IIB 2011 4C7 RANDOM AND NON-LINEAR VIBRATIONS PROF RS LANGLEY / DR AA SESHIA

Q1 a) (i) $R_{xx}(\tau) = \left(\frac{\pi S_0}{\gamma}\right) e^{-\gamma|\tau|} = E[x(t)x(t+\tau)]$

$\Rightarrow E[x^2(t)] = R_{xx}(0) = \underline{\pi S_0/\gamma}$

for the spectrum: $S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{\pi S_0}{\gamma}\right)\left\{\int_{-\infty}^{0} e^{\gamma\tau - i\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-\gamma\tau - i\omega\tau}\, d\tau\right\}$$

note that $|\tau| = -\tau$ for $\tau < 0$

$$= \left(\frac{S_0}{2\gamma}\right)\left\{\left[\frac{1}{\gamma - i\omega}\right]_{-\infty}^{0} - \left[\frac{1}{\gamma + i\omega}\right]_{0}^{\infty}\right\}$$

$$= \left(\frac{S_0}{2\gamma}\right)\left\{\frac{1}{\gamma - i\omega} + \frac{1}{\gamma + i\omega}\right\} = \frac{S_0}{\gamma^2 + \omega^2}$$

$\Rightarrow \underline{S_{xx}(\omega) = \frac{S_0}{\gamma^2 + \omega^2}}$ [20]

(ii) $S_{xx}(\omega) = |H(\omega)|^2 S_{FF}(\omega) \Rightarrow |H(\omega)|^2 = \frac{1}{\gamma^2 + \omega^2} = \left|\frac{1}{\gamma + i\omega}\right|^2$

$\Rightarrow$ expect $H(\omega) = \frac{1}{\gamma + i\omega}$

This is equivalent to $(i\omega + \gamma)\, x(\omega) = F(\omega)$

Fourier transforms of $x(t)$ and $F(t)$

$\Rightarrow \underline{\dot{x} + \gamma x = F(t)}$ – as in question, with $\underline{A = 1,\ B = \gamma}$ [20]

(iii) $\dot{x}(\omega) = i\omega\, x(\omega) \Rightarrow S_{\dot{x}\dot{x}}(\omega) = |i\omega|^2 S_{xx}(\omega) = \omega^2 S_{xx}(\omega)$

$E[\dot{x}^2] = \int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{\omega^2 S_0}{\omega^2 + \gamma^2}\, d\omega$

integrand $\to 1 \times S_0$ as $\omega \to \pm\infty$

$\Rightarrow$ integral is infinity

$E[\dot{x}^2] = \infty \Rightarrow$ Model inadequate to predict $E[\dot{x}^2]$, white noise approximation not realistic.

This question was less popular as indicated by the fewer number of attempts. The first part of the question was generally well done but some students had difficulty with explaining the physical relevance of the results obtained in (a) (iii). Others had difficulty with deriving the condition for finiteness of the mean squared value of the *m*th derivative of *x(t)* in (b).

b) $$a_N \frac{d^N x}{dt^N} + \dots a_0 x = F(t)$$

To find transfer Function, put $x = x(\omega) e^{i\omega t}$, $F(t) = F(\omega) e^{i\omega t}$

$$\Rightarrow \left[a_N (i\omega)^N + a_{N-1} (i\omega)^{N-1} + \dots a_0 \right] x(\omega) = F(\omega)$$

$$\Rightarrow \underline{H(\omega) = \left[\frac{1}{a_N (i\omega)^N + \dots a_0} \right]}$$ - transfer function between x and F

$$\Rightarrow S_{xx}(\omega) = |H(\omega)|^2 S_{FF}(\omega) = |H(\omega)|^2 S_0$$

Now $$FT\left\{ \frac{d^M x}{dt^M} \right\} = (i\omega)^M FT\{x\}$$

$$\Rightarrow S_{x_M x_M}(\omega) = |(i\omega)^M|^2 S_{xx}(\omega) = \omega^{2M} |H(\omega)|^2 S_0$$

x_m representing M'th derivative

$$E[x_M^2] = \int_{-\infty}^{\infty} S_{x_M x_M}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{\omega^{2M} S_0}{|a_N (i\omega)^N + \dots a_0|^2} d\omega$$

Denominator is of order ω^{2N}

Numerator is of order ω^{2M}

$\Rightarrow$ For large ω, integrand is proportional to $\omega^{2(M-N)}$

$\Rightarrow$ Infinite integral for $\underline{M \geq N}$

$\Rightarrow$ $\underline{\text{Finite result only for } M < N}$

White noise approximation is non-sensical for predicting $E\left[\left(\frac{d^N x}{dt^N}\right)^2\right]$

Q2 a) $$M\ddot{x} + C\dot{x} + kx = F(t)$$

$$\ddot{x} + \underbrace{2\beta\omega_n}_{C/M}\dot{x} + \underbrace{\omega_n^2}_{k/M} x = \underbrace{G(t)}_{F/M}$$

F/M, white noise with spectrum $\left(\frac{1}{M}\right)^2 S_0$

Standard results for white noise : $\sigma_{\dot{x}}^2 = \frac{\pi S}{4\beta\omega_n^3} = \frac{\pi S_0/M^2}{2(C/M)(k/M)} = \underline{\frac{\pi S_0}{2Ck}}$

$$\sigma_{\dot{x}}^2 = \omega_n^2 \sigma_x^2 = (k/M)\sigma_x^2 = \underline{\frac{\pi S_0}{2CM}}$$

Standard result for rate of up-crossing b : $\underline{\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2}}$

Probability of impact $= \underline{1 - e^{-\nu_b^+ T}}$ [35%]

b) Probability of impact is governed by ν_b^+. In terms of design variables:

$$\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{k}{M}\right)\exp\left\{-\frac{1}{2}\left[\frac{b^2 2Ck}{\pi S_0}\right]\right\}$$

Now cost $\propto$ $b^3 Ck$

To double cost (for example), double C or k, or multiply b by $2^{1/3}$. Look at the effect of these changes on ν_b^+

$$\nu_b^+ \quad \propto \quad k \quad \exp\left\{-\frac{1}{2}\left[\frac{b^2 2Ck}{\pi S_0}\right]\right\}$$

		k	$\frac{b^2 2Ck}{\pi S_0}$
$k \times 2$	$\Rightarrow$	$\times 2$	$\times 2$
$C \times 2$	$\Rightarrow$	$\times 1$	$\times 2$
$b \times 2^{1/3}$	$\Rightarrow$	$\times 1$	$\times 2^{2/3}$

biggest reduction in ν_b^+ comes from $2 \times C$

$\Rightarrow$ Best design change is to increase damper rate [30%]

c) Standard result $$D = \nu_0^+ T \int_0^\infty \frac{p(S)}{N(S)} dS$$

pdf of stress peaks ($p(S)$); from S-N law ($N(S)$)

For $N = \alpha S^{-\beta}$, standard result $D = \nu_0^+ T \alpha^{-1} (\sqrt{2}\,\sigma_S)^\beta \,\Gamma\left(\frac{\beta+2}{2}\right)$

r.m.s. stress $= k\sigma_x$

So $D \sim \nu_0^+ \sigma_S^{\beta} \sim (k/M)(k\sigma_x)^{\beta} \sim k\left[k\left(\frac{\pi S_0}{2ck}\right)^{\frac{1}{2}}\right]^{\beta} \sim \underline{k^{\beta/2+1} c^{-\beta/2}}$

D is more sensitive to k, but reducing k could increase probability of impact to an unacceptable level. To achieve D target of factor of 2, could:

(i) <u>reduce</u> k by $2^{\frac{1}{\beta/2+1}} = 2^{\frac{2}{\beta+2}}$

(ii) <u>increase</u> C by $2^{2/\beta}$ [35%]

This question was fairly popular as indicated by the large number of attempts. Part (a) was generally well done. Some students had difficulties with formulating the design criteria for reducing impact probability in a cost-efficient way (part (b)). Some students had difficulties with deriving the expression for the fatigue damage in (c).

Q3. (a)

$$\ddot{x} + k_0 x + k_1 x^2 - k_2 x^3 = 0$$

write as:

$$\dot{x} = y$$
$$\dot{y} = -k_0 x - k_1 x^2 + k_2 x^3$$

critical points: $y = 0, \quad x = 0$

and $y = 0, \quad k_0 + k_1 x - k_2 x^2 = 0$

$$y = 0, \quad x = \frac{k_1 \mp \sqrt{k_1^2 + 4 k_0 k_2}}{2k_2}$$

(b)

$$\dot{x} = y$$
$$\dot{y} = k_2 x (x - \alpha_1)(x - \alpha_2)$$

where $\alpha_1 \alpha_2 = -k_0/k_2$ and $\alpha_1 + \alpha_2 = k_1/k_2$

(i) critical point $(0, 0)$:

Linearised matrix $A = \begin{bmatrix} 0 & 1 \\ -k_0 & 0 \end{bmatrix}$

Eigenvalues given by: $\begin{vmatrix} -\lambda & 1 \\ -k_0 & -\lambda \end{vmatrix} = 0$

$$\lambda^2 + k_0 = 0 \quad \text{or} \quad \lambda = \pm i\sqrt{k_0}$$

$\therefore$ $(0, 0)$ is a "centre".

(ii) Linearise about $y = 0, \ x = \alpha_1$

make substitution $x - \alpha_1 = z$

$$\therefore \dot{x} = \dot{z}$$
$$\dot{z} = y$$
$$\dot{y} = k_2 z (z + \alpha_1)(z + \alpha_1 - \alpha_2)$$

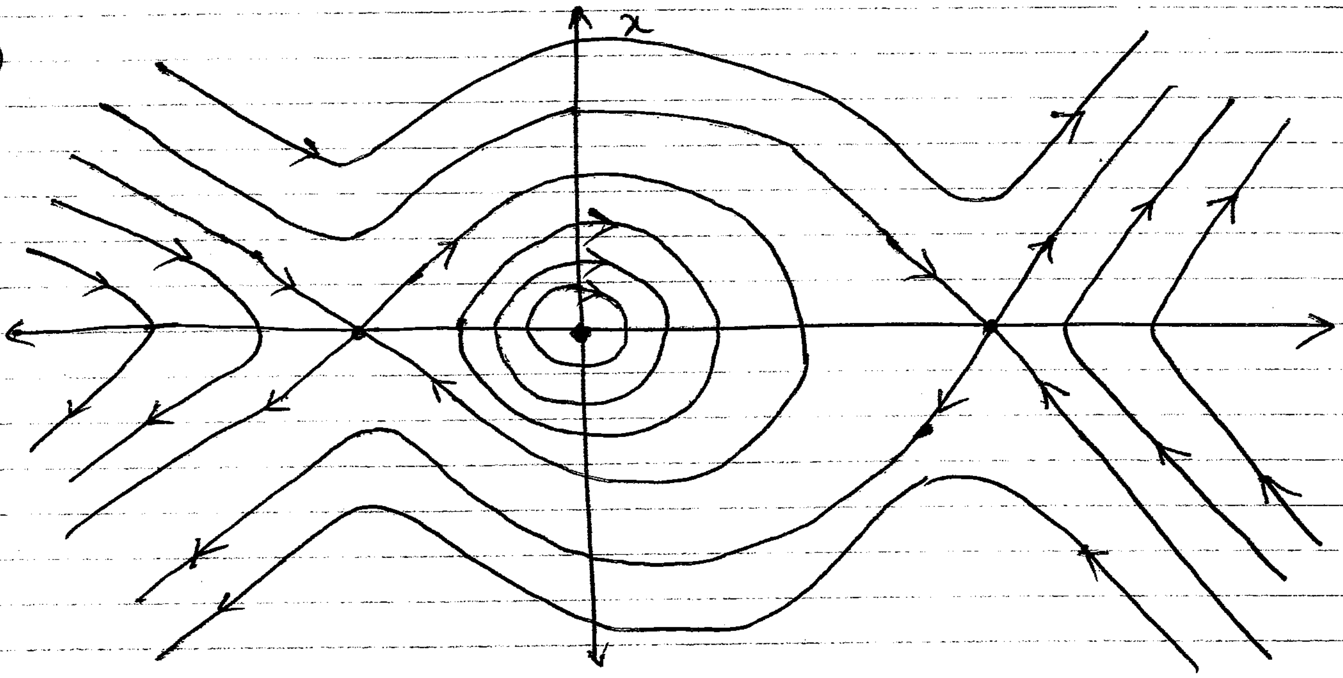
$\therefore$ Linearised matrix $A = \begin{bmatrix} 0 & 1 \\ k_2 \alpha_1 (\alpha_1 - \alpha_2) & 0 \end{bmatrix}$

Eigenvalues of A $\begin{vmatrix} -\lambda & 1 \\ k_2\alpha_1(\alpha_1-\alpha_2) & -\lambda \end{vmatrix} = 0$

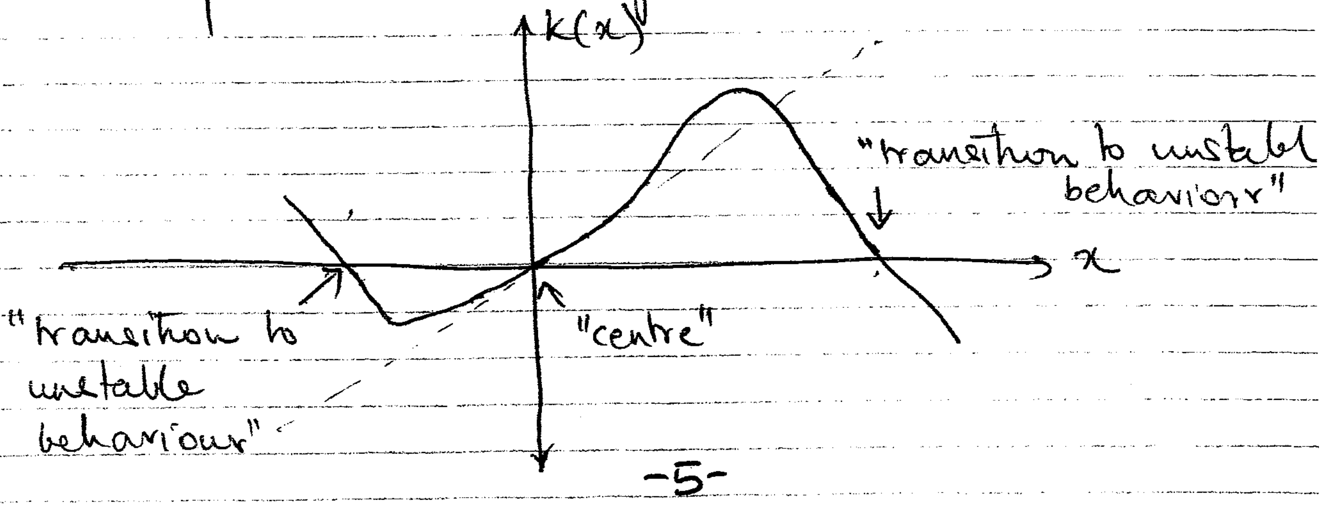
$$\lambda = \pm\sqrt{k_2\alpha_1(\alpha_1-\alpha_2)}$$

If $\alpha_1 > 0$, $\alpha_1-\alpha_2 > 0$ and critical point is a "saddle point"
If $\alpha_1 < 0$, $\alpha_1-\alpha_2 < 0$ and critical point is also a "saddle point".

(c)



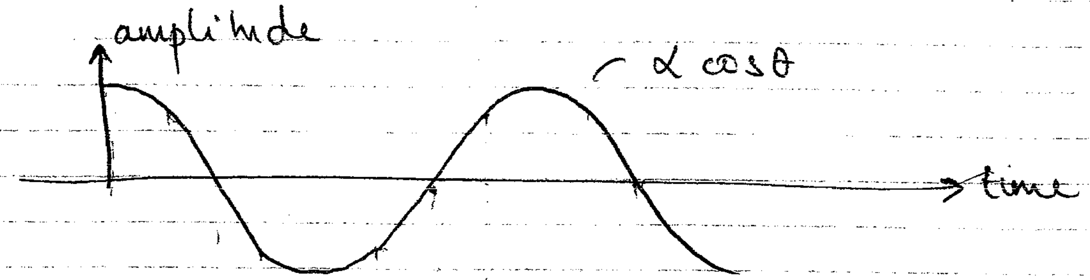
The above phase plane representation makes "physical intuition" in light of the force-displacement diagram below.



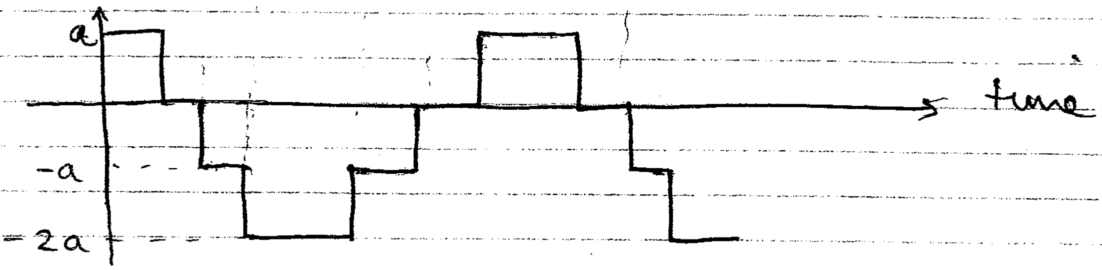
Q3 examiner's comment:

This question was popular as indicated by the large number of attempts. It was also generally well done as indicated by the high average and it appeared that most students had a good grasp of the underlying concepts. Some students had difficulties with proving that two of the three singular points were saddle points. Some students also had difficulties with sketching the restoring force as a function of vibration amplitude in (d).

Q4. (a) Input



Output



(b) Describing Function = Fourier series coefficient of fundamental frequency, divided by input α.

For the case $b < |\alpha| < 2b$:

$$D.F. = \frac{1}{\alpha\pi}\int_0^{2\pi} (\text{output}) \cos\theta \, d\theta$$

$$= \frac{1}{\alpha\pi}\left\{ \int_0^{\pi/2-\beta} a\cos\theta \, d\theta + \int_{\pi/2-\beta}^{\pi/2+\beta} (-a)\cos\theta \, d\theta + \int_{\pi/2+\beta}^{3\pi/2-\beta} (-2a)\cos\theta \, d\theta + \int_{3\pi/2-\beta}^{3\pi/2} (-a)\cos\theta \, d\theta + \int_{3\pi/2+\beta}^{2\pi} (a\cos\theta)\, d\theta \right\}$$

when $\cos\beta = \sqrt{1 - b^2/\alpha^2}$

$$D.F. = \left\{ \sin(\pi/2-\beta) - \left[\sin(\pi/2+\beta) - \sin\pi/2\right] - 2\left[\sin(3\pi/2-\beta) - \sin(\pi/2+\beta)\right] - \left[\sin 3\pi/2 - \sin(3\pi/2-\beta)\right] + \left[\sin(2\pi) - \sin(3\pi/2+\beta)\right] \right\}$$

(b) $\therefore$ D.F. $= \dfrac{a}{\alpha\pi}\left[2 + 2\cos\beta\right]$

$$= \frac{2a}{\alpha\pi}\left[1 + \sqrt{1 - b^2/\alpha^2}\right].$$

For the case $\alpha < b$ then

$$\text{D.F.} = \frac{1}{\alpha\pi}\int_{\pi/2}^{3\pi/2}(-a\cos\theta)\,d\theta = \frac{2a}{\alpha\pi}$$

(c) The equation of motion in the Describing Function approximation is:

$$m\ddot{x} + (\text{D.F.})\,x \simeq a\cos\omega t.$$

if $x = \alpha\cos\omega t$,

$$-m\omega^2\alpha + \frac{2a}{\pi} \simeq a \quad ; \quad \alpha < b$$

$$-m\omega^2\alpha + \frac{2a}{\pi}\left[1 + \sqrt{1 - b^2/\alpha^2}\right] \simeq a \quad ; b < \alpha < 2b$$

In each case we can solve for ω for a specified range of α.

This question was less popular and also was the least well done as indicated by the low average. Many students had difficulty sketching the output waveform for this system and then with deriving the Describing Function in (b).