Version AAS/3

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 27 April 2022 2.00 to 3.40

Module 4C7

RANDOM AND NON-LINEAR VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C7 Random and Non-linear Vibrations data sheet (4 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationary from the Examination Room.

1 A schematic of a vibrational energy harvester is shown in Figure 1. The device is a mass-spring-damper system that is subjected to base acceleration $a(t)$ and a force $F(t)$ applied to the mass. The mass is M, the stiffness is K, and the damping C is provided by an electro-mechanical device that takes energy from the system, so that the power dissipated by the damper can be considered to be harvested. The applied force and acceleration are each zero-mean random processes, having double sided spectral densities $S_{FF}(\omega)$ and $S_{aa}(\omega)$ respectively. The force and the acceleration are uncorrelated and statistically independent.

(a) Show that the spectrum of any quantity of the form $z(t) = F(t) + \alpha a(t)$, is given by $S_{ZZ}(\omega) = S_{FF}(\omega) + \alpha^2 S_{aa}(\omega)$, where α is a constant. [10%]

(b) Write down the equation of motion that governs the relative motion $r(t)$ between the mass and the ground. By assuming that $S_{FF}(\omega)$ and $S_{aa}(\omega)$ can each be approximated as white noise, find an expression for the average harvested power. [40%]

(c) Show that the average power harvested is proportional to M for base acceleration alone ($F = 0$), and inversely proportional to M for applied forcing alone ($a = 0$). Find the value of M for which the harvested power is a minimum in the general case. [20%]

(d) By making a suitable white noise approximation, derive an expression for the mean-squared value of the displacement $x(t)$ of the mass. For a very large value of the stiffness K it might initially be thought that the mass should follow the motion of the ground. Explain why this behaviour is not predicted under the present set of assumptions. Explain why, in reality, the mass *will* actually follow the motion of the ground as $K \to \infty$. [30%]

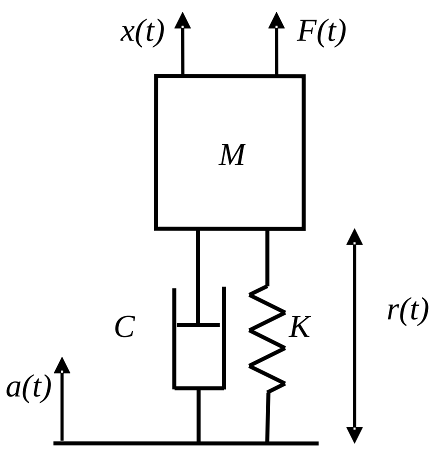

Fig. 1

2 Two circuit boards are mounted in a rack inside an aerospace vehicle. Due to random loads acting on the vehicle the circuit boards undergo random vibrations, and there is a possibility the boards will clash and undergo damage. The centre points of each board are initially separated by a gap of size $2b$, and these points undergo random vibrations $x_1(t)$ and $x_2(t)$ about their mean positions, with the direction and sign convention shown in Fig. 2.

(a) The two boards have the same vibration level so that

$$\sigma_{x_1} = \sigma_{x_2} = \sigma_x, \qquad \sigma_{\dot{x}_1} = \sigma_{\dot{x}_2} = \sigma_{\dot{x}},$$

where σ_x and $\sigma_{\dot{x}}$ are known from measurements. Furthermore, the vibrations are correlated such that

$$\mathrm{E}[x_1 x_2] = \rho\sigma_x^2, \qquad \mathrm{E}[\dot{x}_1 \dot{x}_2] = \rho\sigma_{\dot{x}}^2,$$

where ρ is the correlation coefficient. Derive an expression for the probability that the boards will clash at least once in a time T. Show that the probability is maximised when $\rho = -1$, and that under this condition the probability is equal to the probability that $x_1(t)$ will exceed the value b. Explain this result in physical terms. [35%]

(b) Under the conditions stated in part (a), with $\rho = -1$, find the minimum value of b/σ_x for which the probability of a clash will be less than 10^{-3} for the case $\nu_0^+ T = 5000$, where ν_0^+ is the mean rate at which $x_1(t)$ crosses zero with positive velocity. [30%]

(c) Consider now the case in which the boards have different (known) vibration levels σ_{x_1}, σ_{x_2}, $\sigma_{\dot{x}_1}$ and $\sigma_{\dot{x}_2}$, and the concern is with fatigue damage. If the boards are made on the same material and therefore have the same S-N fatigue law, explain in general terms how you would determine the fatigue life of the system. [35%]

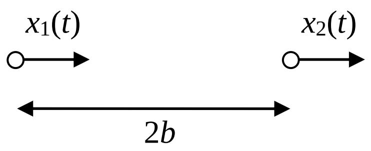

Fig. 2

3 A particle of unit mass is oscillating in a force field with an equation of motion given by

$$\ddot{x} + e^{x} - 1 = 0.$$

(a) Write down an expression for the potential energy of the particle. [10%]

(b) Show that the origin is a singular point. [10%]

(c) Determine the type and stability of the origin as a singular point. [20%]

(d) Sketch the behaviour in the phase plane. [50%]

(e) Write down an equation satisfied by the phase trajectories. [10%]

4 A self-excited system is governed by the following differential equation

$$\ddot{x} + (\alpha - \beta x^2 + \gamma x^4)\dot{x} + x + \mu x^3 = 0,$$

where α, β, γ, μ are positive constants.

(a) By assuming a solution of the form $x = A\cos\Omega t$, find equations for A and Ω by employing the harmonic balance method truncated to the fundamental term. [40%]

(b) Sketch the effective damping constant as a function of amplitude for $\beta^2 > 8\alpha\gamma$. [20%]

(c) Sketch the behaviour of the system in the phase plane for $\beta^2 > 8\alpha\gamma$. [40%]

END OF PAPER

PART IIB Module 4C7
Random and Non-linear Vibrations Data Sheet

Part One: Random Vibration

Gaussian Probability Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{\sigma_y^2(x-\mu_x)^2 + \sigma_x^2(y-\mu_y)^2 - 2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\right\}$$

$$\int_0^\infty e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}, \qquad \int_0^\infty x e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^2, \qquad \int_0^\infty x^2 e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^3\sqrt{\frac{\pi}{2}}$$

Crossing rates: general case

$$\nu_b^+ = \int_0^\infty \dot{x}\, p(b, \dot{x})\mathrm{d}\dot{x}$$

Crossing rates: Gaussian case

$$\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp\left\{-\frac{b^2}{2\sigma^2}\right\}$$

Peak distribution for a narrow band process: general case

$$P(b) = 1 - \frac{\nu_b^+}{\nu_0^+}, \qquad p(b) = -\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{\nu_b^+}{\nu_0^+}\right)$$

Peak distribution for a narrow band process: Gaussian case

$$p(b) = \frac{b}{\sigma_x^2}\exp\left\{-\frac{b^2}{2\sigma_x^2}\right\}$$

Probability of failure after duration T

$$P_f = 1 - \exp\left\{-\nu_b^+ T\right\}$$

Spectral relations

$$S_{xx}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{xx}(\tau)e^{-i\omega\tau}\mathrm{d}\tau, \qquad R_{xx}(\tau)=\int_{-\infty}^{\infty}S_{xx}(\omega)e^{i\omega\tau}\mathrm{d}\omega$$

Input-output relations

$$S_{xx}(\omega)=\left|H(\omega)\right|^2 S_{FF}(\omega)$$

$$S_{x_i x_j}(\omega)=\sum_r\sum_s H_{ir}^*(\omega)H_{js}(\omega)S_{F_rF_s}(\omega)$$

White noise input: standard form

$$m\ddot{x}+c\dot{x}+kx=G, \qquad S_{GG}(\omega)=S_0 \quad \text{(double sided)}$$

$$\sigma_x^2=\frac{\pi S_0}{ck}, \qquad \sigma_{\dot{x}}^2=\frac{\pi S_0}{cm}$$

White noise input: scaled form

$$\ddot{x}+2\beta\omega_n\dot{x}+\omega_n^2 x=F, \qquad S_{FF}(\omega)=S_0 \quad \text{(double sided)}$$

$$\sigma_x^2=\frac{\pi S_0}{2\beta\omega_n^3}, \qquad \sigma_{\dot{x}}^2=\frac{\pi S_0}{2\beta\omega_n}$$

Fatigue Damage

$\mathrm{E}[D]=\mathrm{E}[1/N(S)]\nu_0^+T$ where S is proportional to the peak height b

End of Part One

Part Two: Nonlinear Vibration

Describing functions

System described by the undamped Duffing equation subject to harmonic forcing:

$$\ddot{x} + p^2 x + \mu x^3 = a\cos\omega t$$

$$x \approx \alpha \cos\omega t$$

$$\text{Describing Function} = \frac{output}{input} = \frac{3\mu\alpha^2}{4}$$

Classification of equilibrium or singular points

(nonlinear system)
$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

(linearised about singular point (0,0))
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Eigenvalues of matrix A	**Type of singular point**
Both real and positive	Unstable node
Both real and negative	Stable node
Both real but opposite sign	Saddle point
Both complex with real part = 0	Centre
Both complex with real part > 0	Unstable focus
Both complex with real part < 0	Stable focus

T=tr(**A**) and *D*=det(**A**)

Conditions on *T* and *D*	**Type of singular point**
$D < 0$	Saddle point
$D > 0, T = 0$	Centre
$D > 0, -2\sqrt{D} < T < 0$	Stable focus
$D > 0, 0 < T < 2\sqrt{D}$	Unstable focus
$D > 0, T > 2\sqrt{D}$	Unstable node
$D > 0,\ T < -2\sqrt{D}$	Stable node

Classification of equilibrium or singular points for conservative systems:

(nonlinear system) $$\frac{1}{2}m\dot{x}^2 + V(x) = E$$

Sign of $V''(x)$	**Type of singular point**
positive	Centre
negative	Saddle point

End of Part Two

RSL/AAS
January 2016