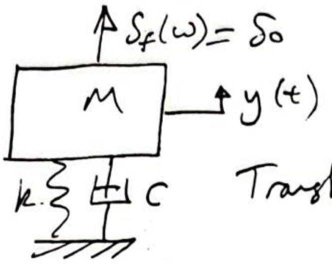
4C7 – 2023

Q1.



$m\ddot{y} + c\dot{y} + ky = f(t)$

Transfer function $H(i\omega) = \frac{Y}{F} = \frac{1}{-m\omega^2 + i\omega c + k}$

Output spectral density: $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

$$\Rightarrow S_y(\omega) = \frac{S_0}{(k - \omega^2 m)^2 + \omega^2 c^2}$$

M.S. Response

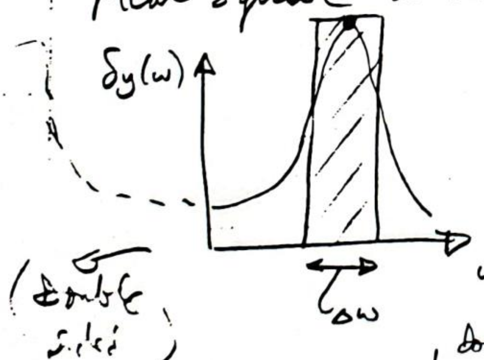
$$E[y^2] = \int_{-\infty}^{\infty} S_y(\omega) = \int_{-\infty}^{\infty} \left| \frac{1}{-m\omega^2 + i\omega c + k} \right|^2 S_0 \, d\omega$$

Using the standard integrals on the data sheet

$B_1 = 0$, $B_0 = \sqrt{S_0}$, $A_2 = m$, $A_1 = c$, $A_0 = k$

$$E[y^2] = \frac{\pi A_0 B_1^2 + A_2 B_0^2}{A_0 A_1 A_2} = \frac{\pi \cancel{m} S_0}{\cancel{m} c k} = \underline{\underline{\frac{\pi S_0}{ck}}}$$

Mean square Bandwidth $\Delta\omega$



Area of rectangle = total area under spectral density

Peak of curve (for light damping set $\omega = \omega_n$

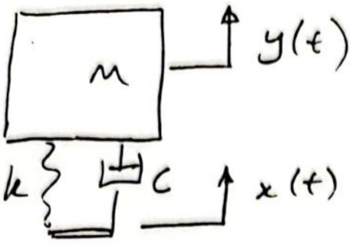
$\Rightarrow |H(\omega_n)| = \frac{1}{\omega_n c}$

$$E[y^2] = \overset{\text{double-sided}}{2} \cdot |H(\omega_n)|^2 S_0 \, \Delta\omega$$

ie $\frac{\pi \cancel{S_0}}{ck} = 2 \left| \frac{1}{\omega_n c} \right|^2 \cancel{S_0} \, \Delta\omega$

$$\Rightarrow \Delta\omega = \frac{\pi}{ck} \frac{c^2}{2} \omega_n^2 = \frac{\pi c}{2m} = \frac{\pi(\zeta \cancel{2}\sqrt{km})}{\cancel{2} m}$$

ie $\underline{\underline{\Delta\omega = \pi \zeta \omega_n}}$

(b) 

$$m\ddot{y} + c\dot{y} + ky = c\dot{x} + kx$$

$$H(i\omega) = \frac{Y}{X} = \frac{k + i\omega c}{m(i\omega)^2 + ci\omega + k}$$

$$\sigma_{\dot{y}}^2 = E[\dot{y}^2] = \int_{-\infty}^{\infty} \omega^2 S_{yy}(\omega)\, d\omega$$

At $\omega = \omega_n$, $H(i\omega_n) \simeq \dfrac{k}{i\omega_n c}$ & $S_{xx}(\omega_n) = \dfrac{S_1}{1 + (\omega_n/\omega_0)^2}$

So $\sigma_{\dot{y}}^2 \simeq 2\omega_n^2 |H(i\omega_n)|^2 S_{xx}(\omega_n)\Delta\omega$

$$= 2\frac{k}{m}\left|\frac{k}{i\omega_n c}\right|^2 \frac{S_1}{1 + (\omega_n/\omega_0)^2}\pi\zeta\omega_n$$

$$= 2\frac{k}{\cancel{m}}\left(\frac{\cancel{m}k}{c^{\cancel{2}}}\right)\frac{S_1}{1 + \frac{k}{m\omega_0^2}}\frac{\pi c}{2m}$$

$$= \underline{\underline{\frac{2k^2\pi S_1}{cm\left(1 + \frac{k}{m\omega_0^2}\right)}}} \quad \text{—— (1)}$$

swap

(b) The probability distribution of peaks for a narrow band process is $p_p(a) = \dfrac{a}{\sigma_{\dot{y}}^2} e^{-a^2/2\sigma_{\dot{y}}^2}$ (data sheet)

So the probability that a peak exceeds a is

$$\int_a^{\infty} p_p(a)\, da = \int_a^{\infty} \frac{a}{\sigma_{\dot{y}}^2} e^{-a^2/2\sigma_{\dot{y}}^2}\, da$$

$$= \left[-e^{-a^2/2\sigma_{\dot{y}}^2}\right]_a^{\infty} = e^{-a^2/2\sigma_{\dot{y}}^2}$$

(d) put $p = e^{-v^2/2\sigma_{\dot{y}}^2} \Rightarrow \sigma_{\dot{y}}^2 = \dfrac{v^2}{2\log_e(1/p)}$

(1) $\rightarrow \dfrac{2k^2\pi S_1}{cm\left(1 + k/m\omega_0^2\right)} = \dfrac{v^2}{2\log_e(1/p)}$ $\therefore$ $c = \underline{\underline{\dfrac{2\pi k^2 S_1 \log_e(1/p)}{mv^2\left(1 + k/m\omega_0^2\right)}}}$

Q2 (a) $R_{yy} = Ae^{-b|\tau|} \qquad -\infty < \tau < \infty$

Using the Fourier Transform relationship from the datasheet :

$$S_{yy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

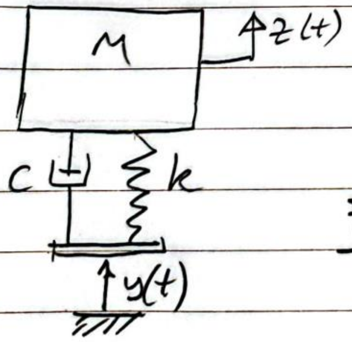
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(A\, e^{-b|\tau|}\right) . e^{-i\omega\tau}\, d\tau$$

$$= \frac{A}{2\pi}\int_{-\infty}^{0} e^{(b-i\omega)\tau}\, d\tau + \frac{A}{2\pi}\int_{0}^{\infty} e^{-(b+i\omega)\tau}\, d\tau$$

$$= \frac{A(1-0)}{2\pi(b-i\omega)} - \frac{A(0-1)}{2\pi(b+i\omega)}$$

$$= \frac{Ab/\pi}{b^2+\omega^2} = \underline{\underline{\frac{A}{b\pi}\left[\frac{1}{1+(\omega/b)^2}\right]}} \quad \text{—— (1)}$$

(b)



$$m\ddot{z} + c\dot{z} + kz = c\dot{y} + ky$$

$$\frac{Z}{Y} = \frac{1 + i2\zeta\,\omega/\omega_n}{1-(\omega/\omega_n)^2 + i2\zeta\,\omega/\omega_n}$$

Dynamic tyre force $= c(\dot{y}-\dot{z}) + k(y-z) = m\ddot{z}$

$$\therefore \frac{F}{Y} = m\frac{\ddot{Z}}{Y} = -\omega^2 m\left(\frac{Z}{Y}\right) = \frac{-\omega^2 m\left(1+i2\zeta\,\omega/\omega_n\right)}{1-\omega^2/\omega_n^2 + i2\zeta\,\omega/\omega_n}$$

$$S_{ff}(\omega) = |H_{F/Y}|^2\, S_{yy}(\omega)$$

2 (b) Cont.

$$S_{ff}(\omega) = \left| \frac{-\omega^2 m\left(1 + i\, 2\zeta \omega/\omega_n\right)}{1 - \omega^2/\omega_n^2 + i\, 2\zeta\, \omega/\omega_n} \right|^2 \cdot \left[\frac{A}{b\pi} \; \frac{1}{1+(\omega/b)^2} \right]$$

$$= \frac{k^2 A}{b\pi} \left[\frac{(\omega/\omega_n)^4 \left(1 + 4\zeta^2 \omega^2/\omega_n^2\right)}{(1-\omega^2/\omega_n^2)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}} \right] \left[\frac{1}{1+(\omega/b)^2} \right]$$

(c) Assuming the process is narrow band, use the M.S. Bandwidth approximation

$$E[f^2] = \int_{-\infty}^{\infty} S_{ff}(\omega)\, d\omega \approx 2 S_{ff}(\omega_n) \cdot \pi \zeta \omega_n$$

$$= \frac{2k^2 A}{b\cancel{\pi}} \left[\frac{1}{4\zeta^2} \right] \left[\frac{1}{1+(\omega_n/b)^2} \right] \cancel{\pi}\, \zeta\, \omega_n$$

$$\therefore \sigma_f^2 = \frac{k^3 A}{bc} \left[\frac{1}{1 + \left(\frac{k}{mb^2}\right)} \right] \qquad [N^2] \quad \text{—— (2)}$$

(d) from data sheet the cumulative probability of $\hat{f}$ for a narrow band process is $P(f_1) = 1 - \frac{\nu_{f_1}^+}{\nu_0^+}$

$$\text{So} \quad Pr(f > f_1) = \frac{\nu_{f_1}^+}{\nu_0^+} = \frac{\frac{1}{2\pi} \frac{\sigma_{\dot{x}}}{\sigma_x} \exp\left(\frac{-f_1^2}{2\sigma_f^2}\right)}{\frac{1}{2\pi}\omega_n}$$

for a narrow band process $E[\dot{f}^2] = \omega_n^2 E[f^2]$

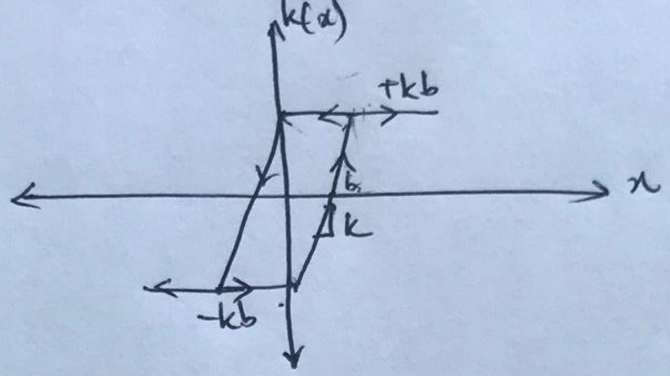
$$\text{so} \quad \frac{\sigma_{\dot{x}}}{\sigma_x} = \omega_n$$

$$\therefore Pr(f > f_1) = \exp\left(-f_1^2/2\sigma_f^2\right)$$

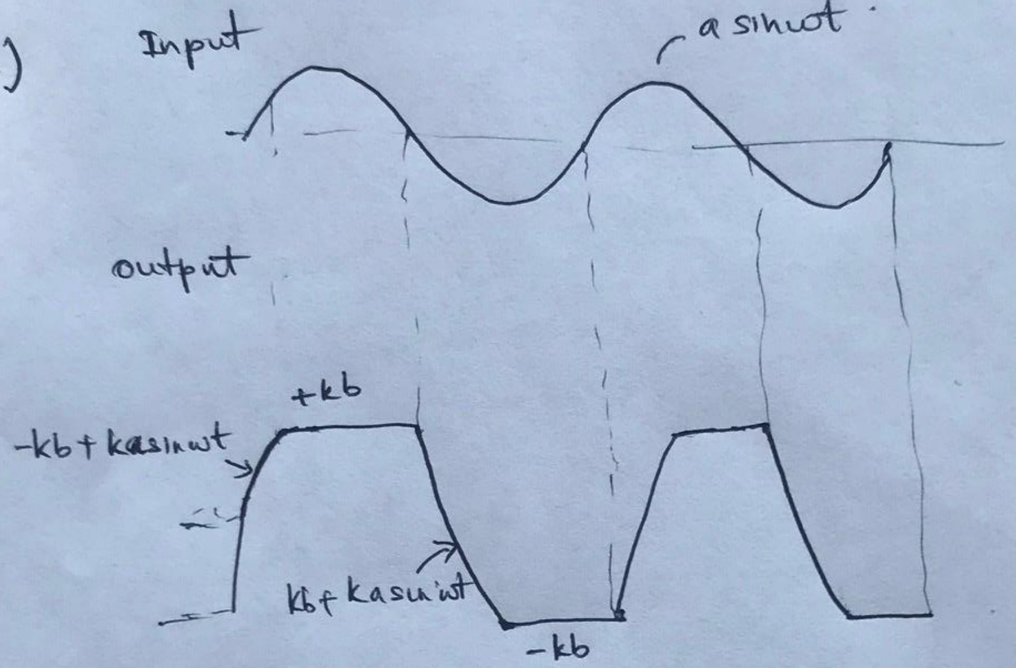
with σ_f from (2)

3.

(a) Maximum value of restoring force when $a > 2b$ $= kb$ in magnitude.



(b)



(c) D.F. given as:

$$\frac{1}{a}(b_1 + j\,a_1)$$

$a\sin\gamma = 2b$

$\gamma = \sin^{-1}(2b/a)$.

where

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} (\text{output}) \sin\theta \, d\theta$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} (\text{output}) \cos\theta \, d\theta .$$

$$\therefore \quad b_1 = \frac{1}{\pi}\left[\int_0^{\gamma} k\sin^2\theta\, d\theta + \int_{\gamma}^{\pi} kb\sin\theta\, d\theta + \int_{\pi}^{\pi+\gamma} k\sin^2\theta\, d\theta \right.$$
$$\left. - \int^{\gamma} kb\sin\theta\, d\theta + \int_{\pi}^{\pi+\gamma} kb\sin\theta\, d\theta - \int^{2\pi} kb\sin\theta\, d\theta \right]$$

(c) $b_1 = \frac{1}{\pi}\left[2\cdot\int_0^{\gamma} k\sin^2\theta\, d\theta + \int_{\gamma}^{\pi} kb\sin\theta\, d\theta - \int_{\pi}^{2\pi} kb\sin\theta\, d\theta \right.$
$\left. -2\int_0^{\gamma} kb\sin\theta\, d\theta + 2\int_{\pi}^{\pi+\gamma} kb\sin\theta\, d\theta\right]$

$= \frac{1}{\pi}\left[2\int_0^{\gamma} k\left(\frac{1-\cos 2\theta}{2}\right)d\theta - kb\left[\cos\theta\right]_{\gamma}^{\pi} + kb\left[\cos\theta\right]_{\pi}^{2\pi}\right.$
$\left. +2kb\left[\cos\theta\right]_0^{\gamma} - 2kb\left[\cos\theta\right]_{\pi}^{\pi+\gamma}\right]$

$= \frac{1}{\pi}\left[2\left[\frac{k\theta}{2}\right]_0^{\gamma} - \frac{2k}{4}\left[\sin 2\theta\right]_0^{\gamma} + 2kb + 2kb\right.$
$\left. + 2kb\left[\cos\gamma - 1\right] - 2kb\left[-\cos\gamma + 1\right]\right]$

$= \frac{1}{\pi}\left[k\gamma - \frac{k}{2}\sin 2\gamma + 4kb\cos\gamma\right].$

$a_1 = \frac{1}{\pi}\left[\int_0^{\gamma}(-kb + k\sin\theta)\cos\theta\, d\theta + \int_{\gamma}^{\pi} kb\cos\theta\, d\theta\right.$
$\left. + \int_{\pi}^{\pi+\gamma}(kb + k\sin\theta)\cos\theta\, d\theta + \int_{\pi+\gamma}^{2\pi}(-kb)\cos\theta\, d\theta\right].$

$= \frac{1}{\pi}\left[\int_0^{\pi} kb\cos\theta\, d\theta - 2\int_0^{\gamma} kb\cos\theta\, d\theta + \int_0^{\gamma}\frac{k\sin 2\theta}{2}d\theta\right.$
$\left. + \int_{\pi}^{\pi+\gamma}\frac{k\sin 2\theta}{2}d\theta - \int_{\pi}^{2\pi} kb\cos\theta\, d\theta + 2\int_{\pi}^{\pi+\gamma} kb\cos\theta\, d\theta\right]$

$= \frac{1}{\pi}\left[kb\left[\sin\theta\right]_0^{\pi} - kb\left[\sin\theta\right]_{\pi}^{2\pi} - 2kb\left[\sin\theta\right]_0^{\gamma}\right.$
$\left. + 2kb\left[\sin\theta\right]_{\pi}^{\pi+\gamma} - \frac{k}{4}\left[\cos 2\theta\right]_0^{\gamma} - \frac{k}{4}\left[\cos 2\theta\right]_{\pi}^{\pi+\gamma}\right]$

$= \frac{1}{\pi}\left[-4kb\sin\gamma - \frac{k}{2}\cos 2\gamma\right]$

$\therefore$ D.F. $= \frac{1}{a}\left[\frac{k\gamma}{\pi} - \frac{k}{2\pi}\sin 2\gamma + \frac{4kb\cos\gamma}{\pi}\right.$
$\left. + \frac{1}{\pi}\left(-4kb\sin\gamma - \frac{k}{2}\cos 2\gamma\right)\right].$

4

(a)
$$\dot{x} = y$$
$$\dot{y} = \epsilon(y - y^3) - x$$

$\dot{x} = \dot{y} = 0$ when $x = 0,\ y = 0$

$\therefore$ The origin is an equilibrium point

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix}$$

Eigenvalues given by (for A)

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \epsilon - \lambda \end{vmatrix} = 0$$

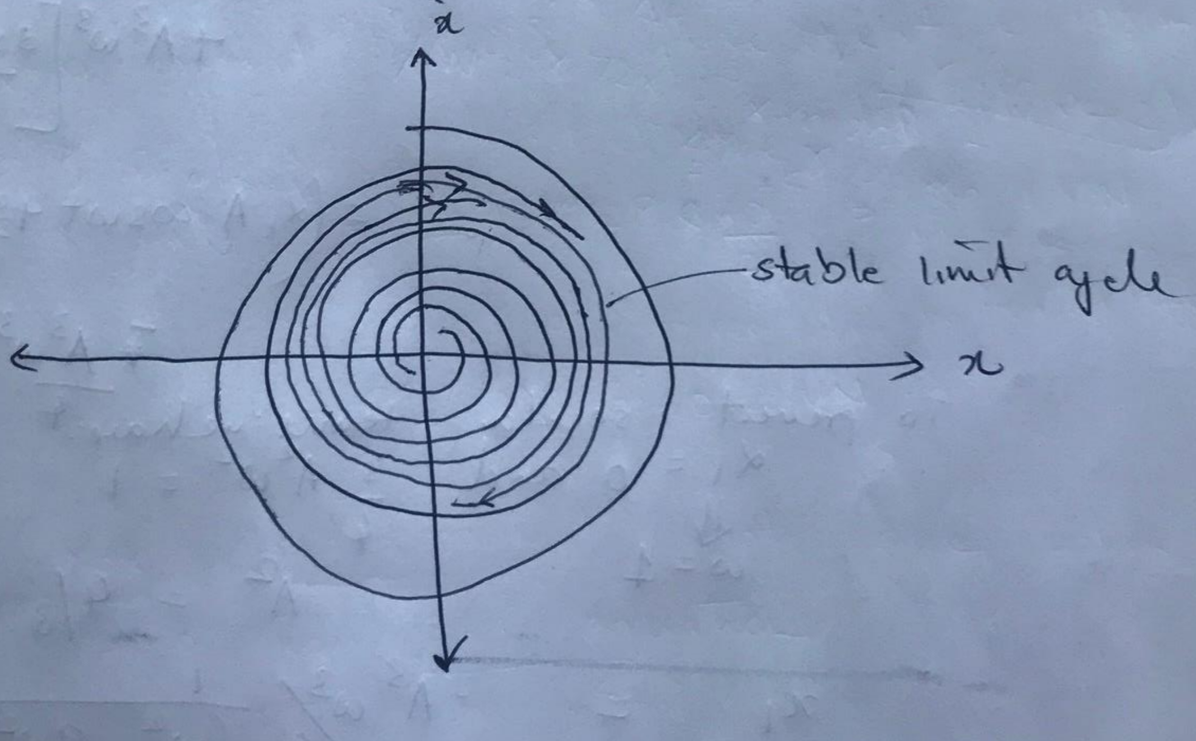
$$\lambda^2 - \epsilon\lambda + 1 = 0$$

$$\lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}$$

Eigenvalues are complex with positive real part

$\therefore$ origin is an unstable focus.

(b)



(c) Use method of perturbation to first order w.r.t.

$$x = x_0 + \epsilon x_1 + \text{H.O.T.}$$

$$\omega^2 = 1 + \epsilon\alpha_1 + \text{H.O.T.}$$

Substituting back into the equation we get:

$$\ddot{x}_0 + \epsilon\ddot{x}_1 - \epsilon(\dot{x}_0 + \epsilon\dot{x}_1 - \dot{x}_0^3) + (\omega^2 - \epsilon\alpha_1)(x_0 + \epsilon x_1) \simeq 0$$

correct to first order in ϵ.

Equating terms in 0th & 1st order in ϵ we get:

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad \text{— (1)}$$

$$\ddot{x}_1 - \dot{x}_0 + \dot{x}_0^3 + \omega^2 x_1 - \alpha_1 x_0 = 0 \quad \text{— (2).}$$

From (1) we have:

$$x_0 = A\cos\omega t \quad \text{say.}$$

From (2):

$$\ddot{x}_1 + \omega^2 x_1 = \alpha_1 A\cos\omega t - A\omega\sin\omega t - A^3\omega^3\sin^3\omega t$$

$$= \alpha_1 A\cos\omega t - A\omega\sin\omega t - \frac{A^3\omega^3}{4}(3\sin\omega t - \sin 3\omega t).$$

Setting terms in $\cos\omega t$ + $\sin\omega t$ to zero on the RHS we get:

$$\alpha_1 = 0 \text{ and } \tfrac{3}{4}A^2\omega^2 = 1.$$

$$\text{or } \omega \sim 1 \text{ and } A \sim 2/\sqrt{3}.$$

$$x_1 \simeq -\frac{A^3\omega^3}{32\omega^2}\sin 3\omega t$$

$$\therefore x \sim 2/\sqrt{3}\cos t - \frac{\epsilon A^3\omega^3}{32\omega^2}\sin 3t.$$