EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 26 April 2023   2.00 to 3.40

Module 4C7

RANDOM AND NON-LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C7 Random and Non-linear Vibrations data sheet (4 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationary from the examination room.

1 The linear oscillator in Fig. 1 has a mass m, supported by a spring of stiffness k and light damping c. The displacement of the oscillator at time t is $y(t)$. The oscillator can be excited by force $f(t)$ or by displacement of its base $x(t)$.

(a) If the displacement input is zero and the force input is white noise, with spectral density:

$$S_{ff}(\omega) = S_0 \qquad -\infty < \omega < \infty,$$

calculate the mean square displacement of the mass $E[y^2]$. Hence show that the *mean square bandwidth* $\Delta\omega$ is given by:

$$\Delta\omega = \pi\zeta\omega_n$$

where ζ is the damping ratio and ω_n is the undamped natural frequency of the oscillator. [30%]

The input force is now set to zero and the base is excited by a displacement $x(t)$ with spectral density

$$S_{xx}(\omega) = \frac{S_1}{1 + (\omega/\omega_0)^2} \qquad -\infty < \omega < \infty.$$

where S_1 and ω_0 are positive, real-valued constants. Since the damping is light, it may be assumed that the response of the oscillator is essentially in a narrow-band of frequencies close to the natural frequency ω_n.

(b) Starting with the Rayleigh distribution for the probability of peaks of a narrow band process exceeding level a, where $a \geq 0$:

$$p_p(a) = \frac{a}{\sigma_y^2} \exp\left\{-\frac{a^2}{2\sigma_y^2}\right\}, \quad a \geq 0$$

show that the probability of a peak exceeding level a is:

$$\mathrm{Prob}(y > a) = \exp\left\{-\frac{a^2}{2\sigma_y^2}\right\}, \quad a \geq 0$$

[20%]

(c) Use the *mean square bandwidth* to estimate the mean square velocity $E[\dot{y}^2]$. [30%]

(d) Using the results of (b) and (c), derive an approximate expression for the amount of viscous damping c required in order to achieve a given value p for the probability that a peak in the velocity $\dot{y}(t)$ exceeds a given level v. [20%]

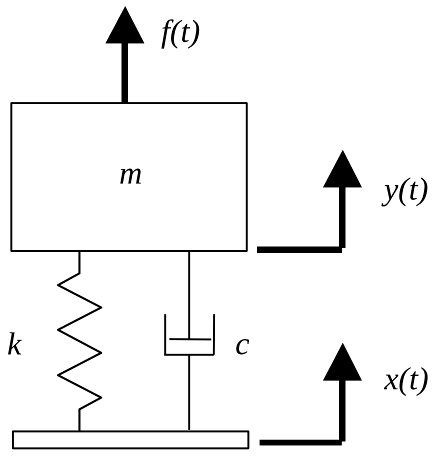

Fig. 1

(TURN OVER

2 Figure 2 shows a simple single degree-of-freedom model of a vehicle which is moving at constant speed over a rough road surface. Point P in the tyre follows displacement profile $y(t)$. The vehicle has mass m and suspension of linear stiffness k and viscous damping c. The displacement of the vehicle body is $z(t)$.

(a) The autocorrelation function of the road profile is measured to be

$$R_{yy}(\tau) = Ae^{-b|\tau|} \qquad -\infty < \tau < \infty$$

where A and b are constants. Show that the road profile spectral density observed by the moving vehicle is

$$S_{yy}(\omega) = \left(\frac{A}{b\pi}\right)\left(\frac{1}{1+(\omega/b)^2}\right)$$ [40%]

(b) Calculate the spectral density $S_{ff}(\omega)$ of the dynamic tyre force (the force in the spring and damper). [20%]

(c) Assuming the dynamic tyre force is a narrow-band process, estimate its mean square value. [20%]

(d) Write an expression for the probability that a peak in the dynamic tyre force exceeds the value f_1. [20%]

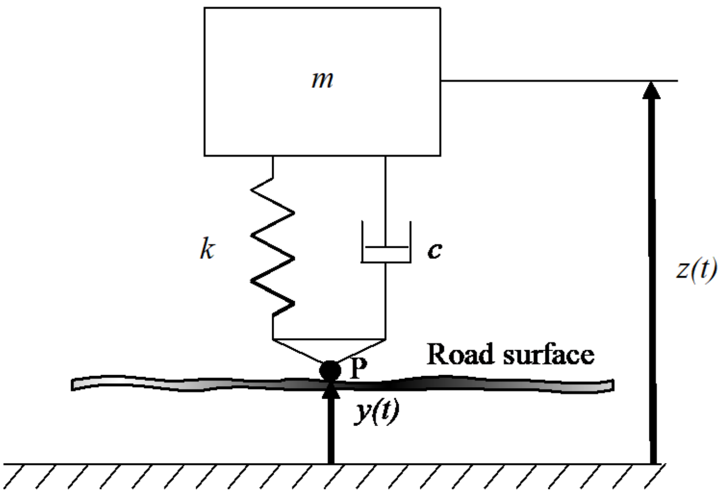

Fig. 2

3 A nonlinear undamped single degree-of-freedom vibratory system of mass m has a symmetrical force-displacement backlash characteristic of width $2b$ and linear response sections of slope κ as shown in Fig. 3. The system is sinusoidally driven at an angular frequency ω and the response amplitude is a.

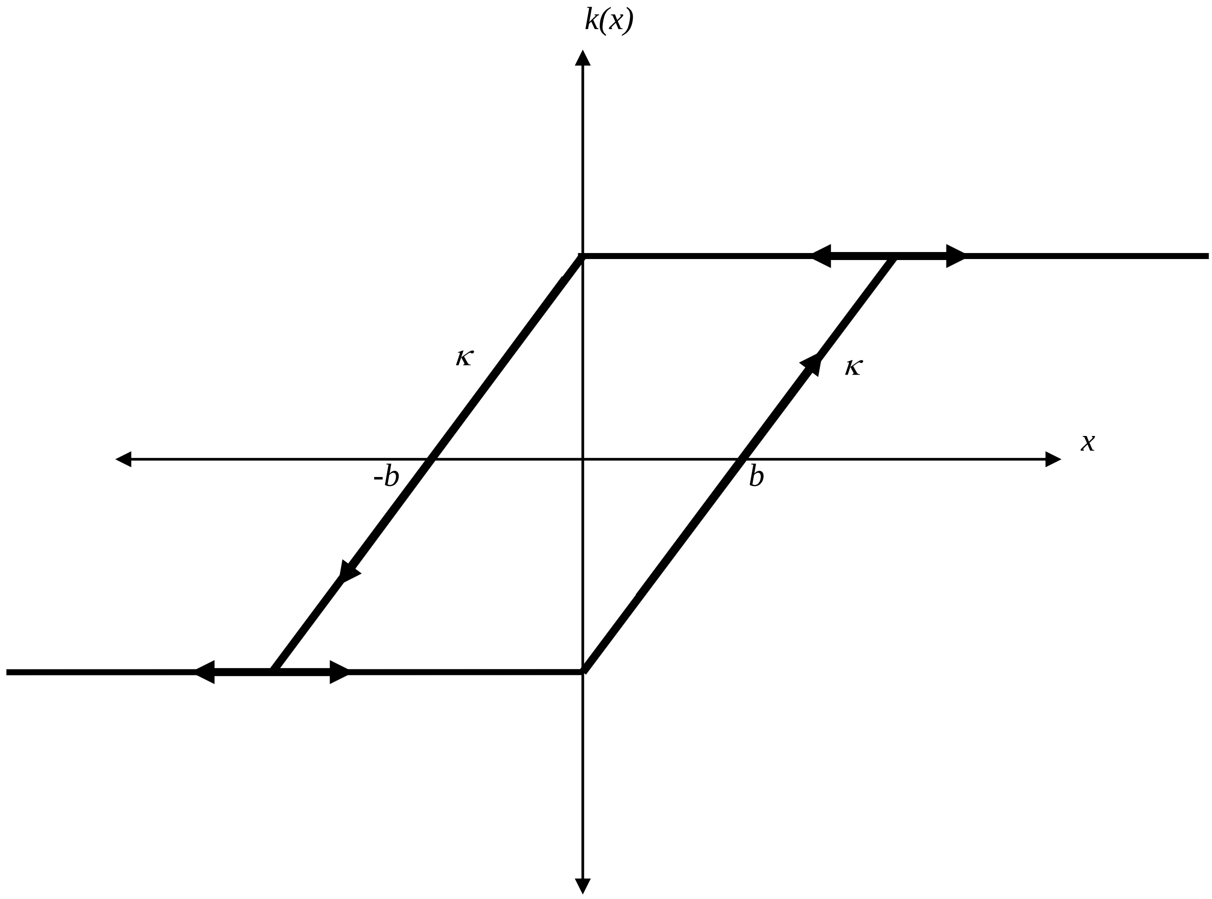

Fig. 3

(a) Determine the maximum value of the restoring force in terms of the parameters shown. [10%]

(b) Sketch the input and output waveforms when the system is sinusoidally driven at an angular frequency ω with a response amplitude $a > 2b$. [30%]

(c) Determine the Describing Function for a sinusoidal input corresponding to a response amplitude $a > 2b$. [60%]

4 A nonlinear system is characterised by the equation below where ε can be considered a small positive parameter ($0 < \varepsilon << 1$):

$$\ddot{x} - \varepsilon\left(\dot{x} - \dot{x}^3\right) + x = 0 \,.$$

(a) Show that the origin is an equilibrium point and determine its type and stability. [20%]

(b) Sketch the behaviour of the system in the phase plane showing the limit cycle. [20%]

(c) Use the method of perturbation to obtain an estimate of the steady-state amplitude of the limit cycle response to first order in ε. [60%]

END OF PAPER

PART IIB Module 4C7
Random and Non-linear Vibrations Data Sheet

Part One: Random Vibration

Gaussian Probability Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{\sigma_y^2(x-\mu_x)^2 + \sigma_x^2(y-\mu_y)^2 - 2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\right\}$$

$$\int_0^\infty e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}, \qquad \int_0^\infty x e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^2, \qquad \int_0^\infty x^2 e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^3\sqrt{\frac{\pi}{2}}$$

Crossing rates: general case

$$\nu_b^+ = \int_0^\infty \dot{x} p(b, \dot{x})\mathrm{d}\dot{x}$$

Crossing rates: Gaussian case

$$\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp\left\{-\frac{b^2}{2\sigma^2}\right\}$$

Peak distribution for a narrow band process: general case

$$P(b) = 1 - \frac{\nu_b^+}{\nu_0^+}, \qquad p(b) = -\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{\nu_b^+}{\nu_0^+}\right)$$

Peak distribution for a narrow band process: Gaussian case

$$p(b) = \frac{b}{\sigma_x^2}\exp\left\{-\frac{b^2}{2\sigma_x^2}\right\}$$

Probability of failure after duration T

$$P_f = 1 - \exp\left\{-\nu_b^+ T\right\}$$

Spectral relations

$$S_{xx}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{xx}(\tau)e^{-i\omega\tau}\mathrm{d}\tau, \qquad R_{xx}(\tau)=\int_{-\infty}^{\infty}S_{xx}(\omega)e^{i\omega\tau}\mathrm{d}\omega$$

Input-output relations

$$S_{xx}(\omega)=\left|H(\omega)\right|^2 S_{FF}(\omega)$$

$$S_{x_i x_j}(\omega)=\sum_r\sum_s H_{ir}^*(\omega)H_{js}(\omega)S_{F_rF_s}(\omega)$$

Calculation of mean-square response integrals

$$I_n=\int_{-\infty}^{\infty}\left|H_n(i\omega)\right|^2 d\omega$$

where $H_n(i\omega)=\dfrac{B_0+(i\omega)B_1+\ldots+(i\omega)^{n-1}B_{n-1}}{A_0+(i\omega)A_1+\ldots+(i\omega)^n A_n}$,

and the roots λ of the characteristic equation $\left(A_0+\lambda A_1+\lambda^2 A_2+\ldots+\lambda^n A_n=0\right)$ all have negative real parts, takes the values:

$$I_1=\pi\frac{B_0^2}{A_0A_1} \qquad I_2=\pi\frac{A_0B_1^2+A_2B_0^2}{A_0A_1A_2}$$

$$I_3=\pi\frac{A_0A_1B_2^2+A_2A_3B_0^2+A_0A_3\left(B_1^2-2B_0B_2\right)}{A_0A_3\left(A_1A_2-A_0A_3\right)}.$$

White noise input: standard form

$$m\ddot{x}+c\dot{x}+kx=G, \qquad S_{GG}(\omega)=S_0 \quad \text{(double sided)}$$

$$\sigma_x^2=\frac{\pi S_0}{ck}, \qquad \sigma_{\dot{x}}^2=\frac{\pi S_0}{cm}$$

White noise input: scaled form

$$\ddot{x}+2\beta\omega_n\dot{x}+\omega_n^2 x=F, \qquad S_{FF}(\omega)=S_0 \quad \text{(double sided)}$$

$$\sigma_x^2=\frac{\pi S_0}{2\beta\omega_n^3}, \qquad \sigma_{\dot{x}}^2=\frac{\pi S_0}{2\beta\omega_n}$$

Fatigue Damage

$\mathrm{E}[D]=\mathrm{E}[1/N(S)]\nu_0^+T$ where S is proportional to the peak height b

End of Part One

Part Two: Nonlinear Vibration

Describing functions

System described by the undamped Duffing equation subject to harmonic forcing:

$$\ddot{x} + p^2 x + \mu x^3 = a\cos\omega t$$

$$x \approx \alpha \cos\omega t$$

$$\text{Describing Function} = \frac{output}{input} = \frac{3\mu\alpha^2}{4}$$

Classification of equilibrium or singular points

(nonlinear system)
$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

(linearised about singular point (0,0))
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Eigenvalues of matrix A	**Type of singular point**
Both real and positive	Unstable node
Both real and negative	Stable node
Both real but opposite sign	Saddle point
Both complex with real part = 0	Centre
Both complex with real part > 0	Unstable focus
Both complex with real part < 0	Stable focus

T=tr(**A**) and D=det(**A**)

Conditions on *T* and *D*	**Type of singular point**
$D < 0$	Saddle point
$D > 0, T = 0$	Centre
$D > 0, -2\sqrt{D} < T < 0$	Stable focus
$D > 0, 0 < T < 2\sqrt{D}$	Unstable focus
$D > 0, T > 2\sqrt{D}$	Unstable node
$D > 0,\ T < -2\sqrt{D}$	Stable node

Classification of equilibrium or singular points for conservative systems:

(nonlinear system) $$\frac{1}{2}m\dot{x}^2 + V(x) = E$$

Sign of $V''(x)$	**Type of singular point**
positive	Centre
negative	Saddle point

End of Part Two

RSL/AAS/DC
December 2020