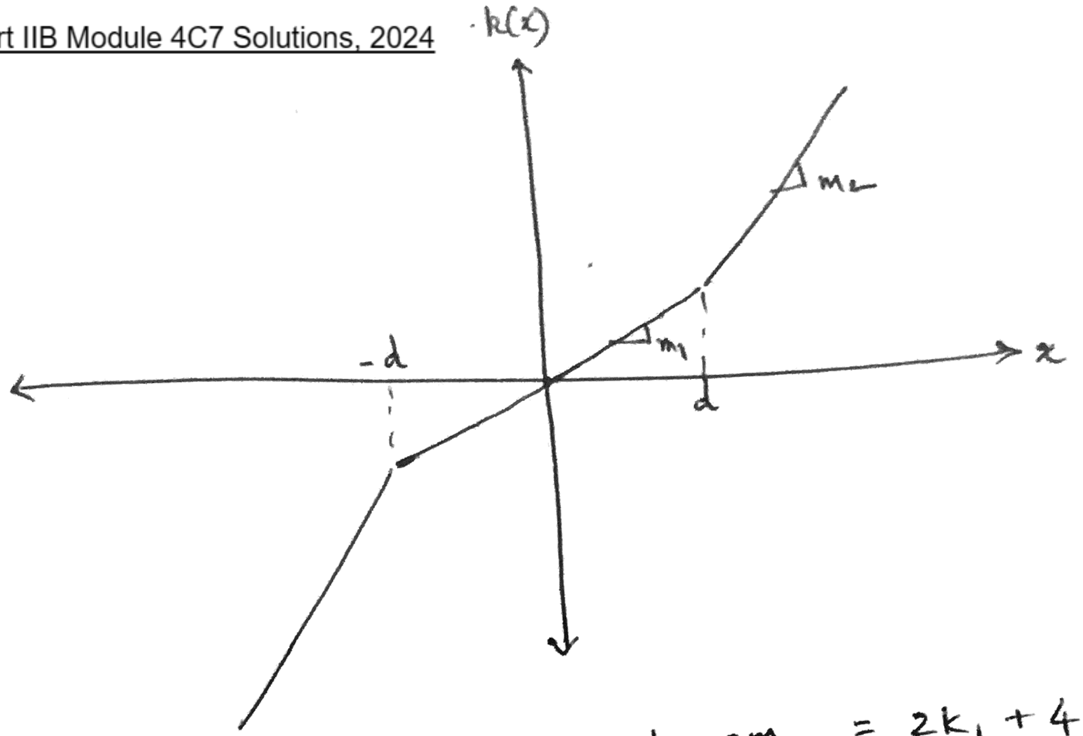
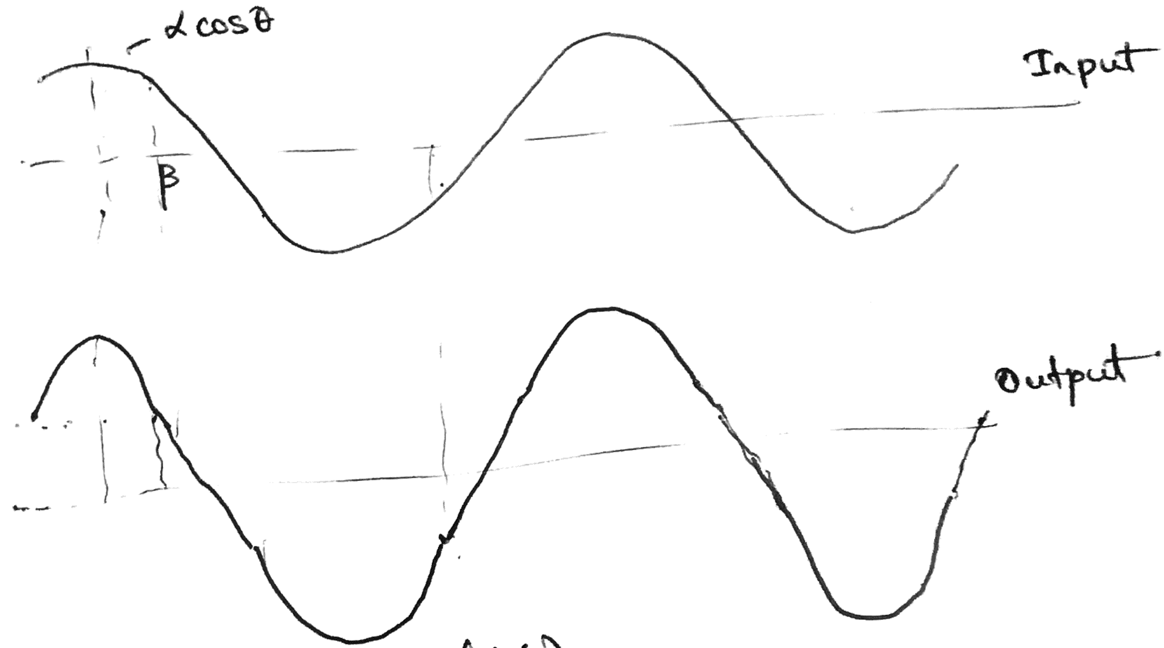
(a)



slope $m_1 = 2k_1$, slope $m_2 = 2k_1 + 4k_2$

(b)



$$\beta = \cos^{-1}\left(d/\alpha\right).$$

(c)

$$D.F. = \frac{1}{\alpha\pi}\int_0^{2\pi} (\text{output}) \times \cos\theta \, d\theta$$

$$= \frac{4}{\alpha\pi}\left[\int_0^{\beta}\left[(m_1 - m_2)\delta + m_2\alpha\cos\theta\right]\cos\theta\, d\theta + \int_{\beta}^{\pi/2} m\alpha\cos^2\theta \, d\theta\right]$$

$$D.F. = \frac{4}{\pi\alpha}\left(\left[(m_1-m_2)\delta\sin\theta + \frac{m_2\alpha\theta}{2} + \frac{m_2\alpha\sin 2\theta}{4}\right]_0 + \left[\frac{m_1\alpha\theta}{2} + \frac{m_1\alpha\sin 2\theta}{4}\right]_\beta^{\pi/2}\right)$$

$$D.F. = \frac{4}{\pi\alpha}\left((m_1-m_2)\delta\sin\beta + \frac{m_2\alpha\beta}{2} + \frac{m_2\alpha\sin\beta\cos\beta}{2} + \frac{m_1\alpha\pi}{4} - \frac{m_1\alpha\beta}{2} \mp \frac{m_1\alpha\sin\beta\cos\beta}{2}\right)$$

$$D.F. = \frac{4}{\pi\alpha}\left((m_1-m_2)\left[\delta\sin\beta - \frac{\alpha}{2}\sin\beta\cos\beta - \frac{\alpha}{2}\beta\right]\right) + m_1$$

$$= \frac{2(m_1-m_2)}{\pi}\left(\frac{2\delta}{\alpha}\sqrt{1-\frac{\delta^2}{\alpha^2}} - d/\alpha\sqrt{1-d^2/\alpha^2} - \cos^{-1}(d/\alpha)\right) + m_1.$$

(d) The equation to determine response amplitude α over a range of frequencies can be written as:

$$-m\omega^2\alpha + D.F. \times \alpha \simeq af.$$

for $a > d$.

(a)

$$\dot{x} = y$$
$$\dot{y} = (\beta - \delta y^2)y + \alpha x - \gamma x^3$$

Equilibrium points given by:

$$\dot{x} = \dot{y} = 0.$$

$$\therefore \dot{x} = 0, \quad \alpha x - \gamma x^3 = 0.$$

$$\dot{x} = 0, \quad x = 0 \ \& \ x = \pm\sqrt{\alpha/\gamma}$$

For equilibrium point $(0,0)$

$$A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}$$

eigenvalues given by:

$$\begin{vmatrix} -\lambda & 1 \\ \alpha & \beta - \lambda \end{vmatrix} = 0.$$

$$\lambda^2 - \beta\lambda - \alpha = 0.$$

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$$

Eigenvalues are real and opposite sign and therefore a saddle point.

For equilibrium point $x = \sqrt{\alpha/\gamma}$ define

$$z = \left(x - \sqrt{\frac{\alpha}{\gamma}}\right)$$

equations now are:

$$\dot{z} = y$$
$$\dot{y} = (\beta - \delta y^2)y + \left(z + \sqrt{\frac{\alpha}{\gamma}}\right)$$

$$A = \begin{bmatrix} 0 & 1 \\ -2\gamma\alpha & \beta \end{bmatrix}$$

Eigenvalues are given by:

$$\begin{vmatrix} -\lambda & 1 \\ -2\gamma\alpha & \beta-\lambda \end{vmatrix} = 0.$$

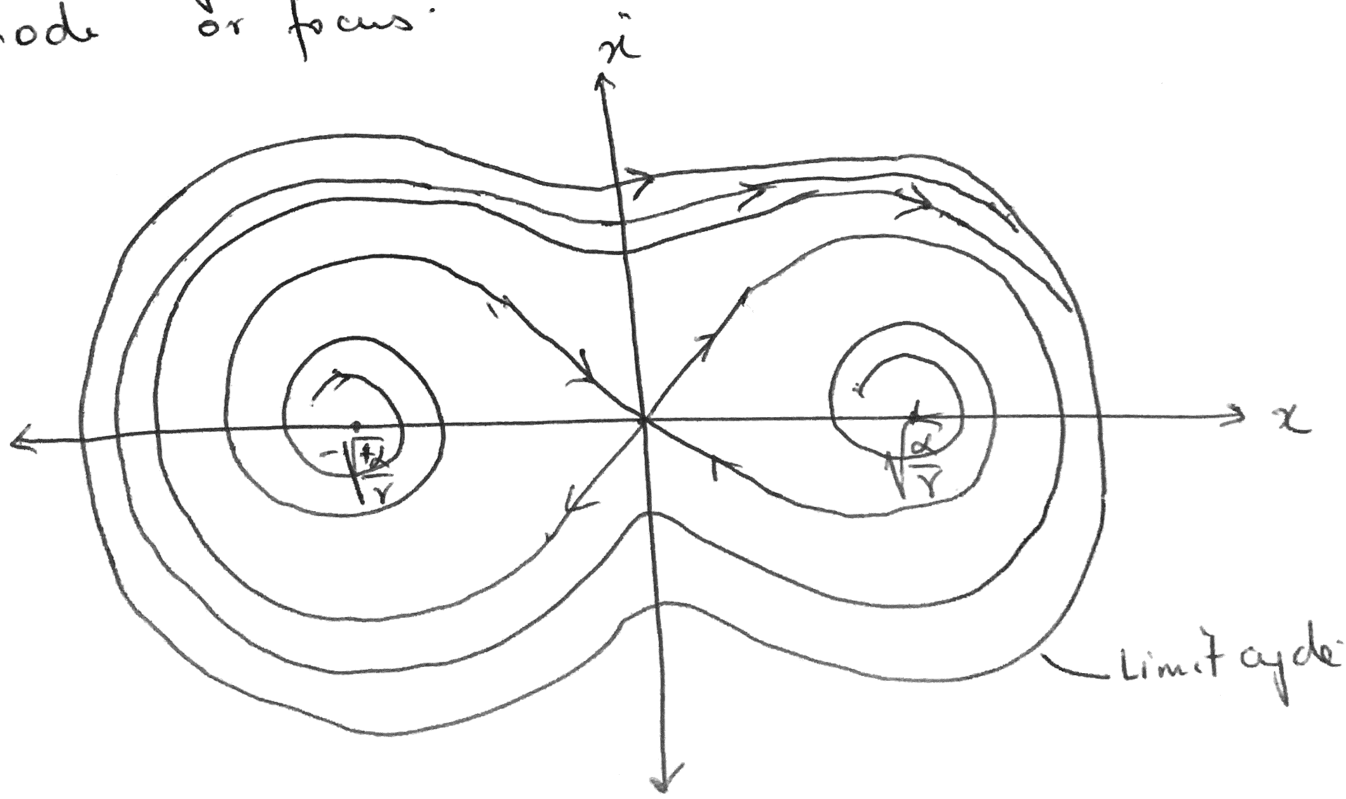
$$\lambda^2 - \beta\lambda + 2\gamma\alpha = 0.$$

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 - 8\gamma\alpha}}{2}$$

Two possibilities exist. Either $\beta^2 - 8\alpha < 0$ or $\beta^2 - 8\alpha > 0$

equilibrium point is an unstable node or focus.

For the third equilibrium point we have similarly to the second either an unstable node or focus.



For the case where the equilibrium points at $x = \pm\sqrt{\frac{\alpha}{\gamma}}$ are unstable foci

$$-A\Omega^2 - \alpha A + \frac{3\gamma A^3}{4} = 0 \qquad \text{— (1)}$$

$$\beta A\Omega - \frac{\delta A^3\Omega^3}{2} - \frac{\delta A^3\Omega^3}{4} = 0 \qquad \text{— (2)}$$

From (2) assuming $\Omega \neq 0$ we get

$$\frac{3}{4}\delta A^3\Omega^2 = \beta A$$

or

$$\Omega^2 = \frac{4}{3}\frac{\beta}{\delta A^2} \quad \text{for } A \neq 0 \qquad \text{— (3).}$$

substituting (3) in (1) we get:

$$-\frac{4}{3}\frac{\beta}{\delta A} - \alpha A + \frac{3\gamma A^3}{4} = 0.$$

or

$$\frac{9\delta\gamma A^4}{4} - 3\delta\alpha A^2 - 4\beta = 0.$$

$$A^2 = \left(\frac{3\delta\alpha + \sqrt{9\delta\alpha^2 + 36\delta\gamma\beta}}{9\delta\gamma}\right) \times 2$$

$$= \frac{2\delta\alpha + 2\sqrt{\delta\alpha^2 + 4\delta\gamma\beta}}{3\delta\gamma}$$

picking the root with physical meaning.

3 (a) As there is no delta function in $S_s(\omega)$ at $\omega=0$, $M_s=0$

Hence $\sigma_s^2 = E[s^2] = \int_{-\infty}^{\infty} S_s(\omega)\,d\omega = \underline{\underline{2S_0\Delta\omega}}$

$$\sigma_{\dot{s}}^2 = \int_{-\infty}^{\infty} \omega^2 S_s(\omega)\,d\omega = \underline{\underline{2S_0\omega_0^2\Delta\omega}}$$

(b) Assuming s & $\dot{s}$ are independent Gaussian processes

$$p(s,\dot{s}) = \frac{1}{2\pi\sigma_s\sigma_{\dot{s}}} e^{-s^2/2\sigma_s^2} e^{-\dot{s}^2/2\sigma_{\dot{s}}^2}$$

$$\therefore \nu_a^+ = \int_0^\infty \frac{e^{-a^2/2\sigma_s^2}}{2\pi\sigma_s\sigma_{\dot{s}}}\dot{s}e^{-\dot{s}^2/2\sigma_{\dot{s}}^2}\,d\dot{s} = \frac{e^{-a^2/2\sigma_s^2}}{2\pi\sigma_s\sigma_{\dot{s}}}\underbrace{\int_0^\infty \dot{s}e^{-\dot{s}^2/2\sigma_{\dot{s}}^2}\,d\dot{s}}_{\sigma_{\dot{s}}^2 \text{ (data sheet)}}$$

$$= \frac{e^{-a^2/2\sigma_s^2}}{2\pi\sigma_s\sigma_{\dot{s}}}\sigma_{\dot{s}}^2 = \frac{1}{2\pi}\frac{\sigma_{\dot{s}}}{\sigma_s}e^{-a^2/2\sigma_s^2}$$

Substituting for σ_s & $\sigma_{\dot{s}}$ from (a) gives

$$\underline{\underline{\nu_a^+ = \frac{1}{2\pi}\omega_0 e^{-a^2/4S_0\Delta\omega}}}$$

(c) Prob (peak $>a$) = proportion of peaks $>a$

$$= \int_a^\infty p_p(s)\,ds = \frac{\nu_a^+}{\nu_0^+} = e^{-a^2/4S_0\Delta\omega}$$

$$\therefore p_p(s) = -\frac{d}{da}\left(e^{-a^2/4S_0\Delta\omega}\right)\Big|_{a=s} = \underline{\underline{\frac{s}{2S_0\Delta\omega}e^{-s^2/4S_0\Delta\omega}}}$$

(d) Expected number of cycles with peaks in the range s to $s+ds$ in time T

= (Total number of cycles in T) $\times$ Prob ($s <$ peak $< s+ds$)

$$= \nu_0^+ T \times p_p(s)\,ds = \underline{\underline{\frac{s\omega_0 T}{4\pi S_0\Delta\omega}e^{-s^2/4S_0\Delta\omega}\,ds}}$$

1(e) Number of cycles to failure at stress level s is $N_f(s) = C s^{-k}$

Total number of cycles at this level in time T is $\nu_0^+ T\, p_p(s)\, ds$

$\therefore$ failure occurs (according to Miner's rule) when:

$$\int_0^\infty \frac{\nu_0^+ T\, p_p(s)\, ds}{N_f(s)} = 1$$

$$\Rightarrow \int_0^\infty \frac{\omega_0 T}{4\pi S_0 \Delta\omega C} s^{k+1} e^{-s^2/4S_0\Delta\omega}\, ds = 1$$

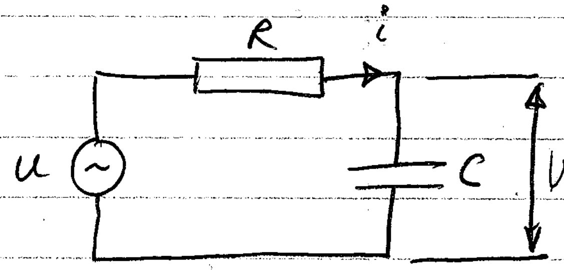
$$\therefore T = \underline{\underline{\left[\int_0^\infty \frac{\omega_0}{4\pi S_0 \Delta\omega C} s^{k+1} e^{-s^2/4S_0\Delta\omega}\, ds\right]^{-1}}}$$

for $k=1$ $\quad T = \left[\frac{\omega_0}{4\pi S_0 \Delta\omega C} \underbrace{\int_0^\infty s^2 e^{-s^2/2\sigma_s^2}\, ds}_{\sigma_s^3\sqrt{\pi/2} \text{ (data sheet)}}\right]^{-1}$

$$\text{So } T = \left[\frac{\omega_0}{4\pi S_0 \Delta\omega C}\sqrt{\frac{\pi}{2}}\,(2S_0\Delta\omega)^{3/2}\right]^{-1}$$

$$= \underline{\underline{\frac{2\sqrt{\pi}\,C}{\omega_0\sqrt{S_0\Delta\omega}}}}$$

4/(a)



Voltages : $u = IR + V$ ——— (1)

Capacitor : $i = C\dot{V}$ ——— (2)

(2) into (1) : $RC\dot{V} + V = u$ ——— (3)

Set $v = Ve^{j\omega t}$ & $u = Ue^{j\omega t}$

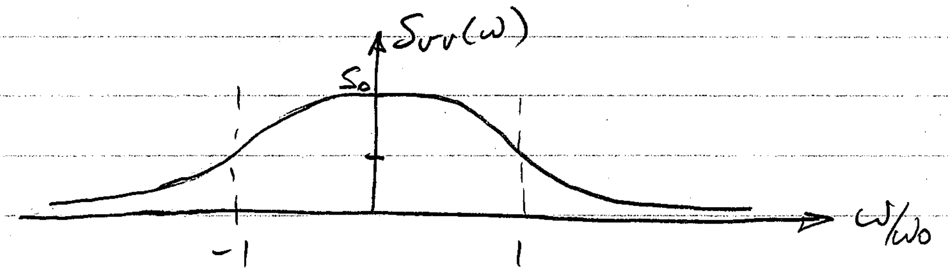
gives $V(1 + j\omega RC) = U$

ie $\dfrac{V}{U} = \dfrac{1}{1 + j\omega RC} = \dfrac{1}{1 + j\,\omega/\omega_0}$ ——— (4)

where $\omega_0 = 1/RC$

(b)(i) If u is white noise so that $S_{uu}(\omega) = S_0 \quad -\infty < \omega < \infty$

then $S_{vv}(\omega) = \left|\dfrac{1}{1 + j\,\omega/\omega_0}\right|^2 S_0 = \dfrac{S_0}{1 + (\omega/\omega_0)^2}$ ——— (5)



(ii) $\sigma_v^2 = \displaystyle\int_{-\infty}^{\infty} S_{vv}(\omega)\,d\omega = 2\int_0^{\infty} \frac{S_0}{1 + (\omega/\omega_0)^2}\,d\omega$

$= 2S_0\omega_0\left[\tan^{-1}\omega/\omega_0\right]_0^{\infty} = \pi S_0 \omega_0$

—1—

(iii) $R_{vv}(\tau) = \pi S_0 \omega_0 \, e^{-\omega_0|\tau|}$

Check $R_{vv}(0) = \sigma^2 = \pi S_0 \omega_0$ ✓

<u>Verify</u>: $S_{uu}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_v(\tau)\, e^{-j\omega\tau}\, d\tau$ (Fourier Transform)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi S_0 \omega_0 \, e^{-(\omega_0|\tau| + j\omega\tau)}\, d\tau$$

$$= \frac{S_0\omega_0}{2}\int_{-\infty}^{0} e^{\omega_0\tau - j\omega\tau}\, d\tau + \frac{S_0\omega}{2}\int_{0}^{\infty} e^{-\omega_0\tau - j\omega\tau}\, d\tau$$

$$= \frac{S_0\omega_0}{2}\left\{\left[\frac{e^{(\omega_0 - j\omega)\tau}}{\omega_0 - j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\omega_0 + j\omega)\tau}}{-(\omega_0 + j\omega)}\right]_{0}^{\infty}\right\}$$

$$= \frac{S_0\omega_0}{2}\left\{\frac{1}{\omega_0 - j\omega} + \frac{1}{\omega_0 + j\omega}\right\}$$

$$= \frac{S_0\omega_0^2}{\omega_0^2 + \omega^2} = \underline{\underline{\frac{S_0}{1 + (\omega/\omega_0)^2}}}$$ as per (5) ✓

(c) Using (2) with $i(t) = I e^{j\omega t}$ gives $I = j\omega C V$ (6)

(4) & (6) $\Rightarrow \frac{I}{u} = \frac{j\omega C}{1 + j\omega RC} = \frac{1}{R}\left[\frac{j\,\omega/\omega_0}{1 + j\,\omega/\omega_0}\right]$

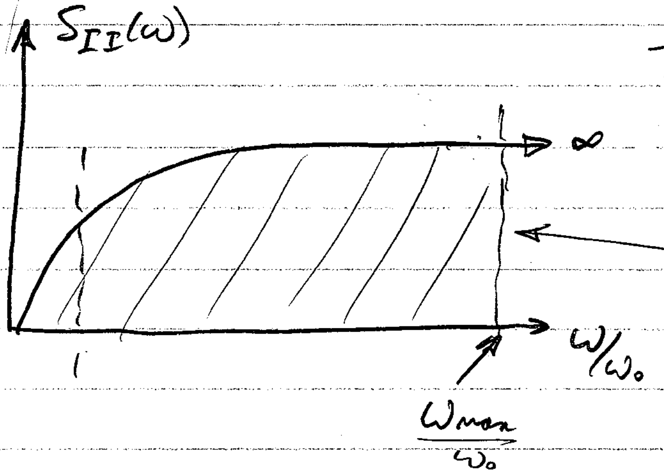
With $S_{uu}(\omega) = S_0$,

$$S_{II}(\omega) = \frac{S_0}{R^2}\left|\frac{j\,\omega/\omega_0}{1 + j\,\omega/\omega_0}\right|^2 = \frac{S_0}{R^2}\left[\frac{(\omega/\omega_0)^2}{1 + (\omega/\omega_0)^2}\right]$$

Power $= i^2 R \longrightarrow \overline{P} = R\,E[i^2] = R\sigma_i^2$

So $\overline{P} = R\int_{-\infty}^{\infty} S_{II}(\omega)\, d\omega = \frac{2S_0}{R}\int_0^{\infty} \frac{(\omega/\omega_0)^2}{1 + (\omega/\omega_0)^2}\, d\omega$ (7)

(d) The problem with (7) is that it doesn't converge

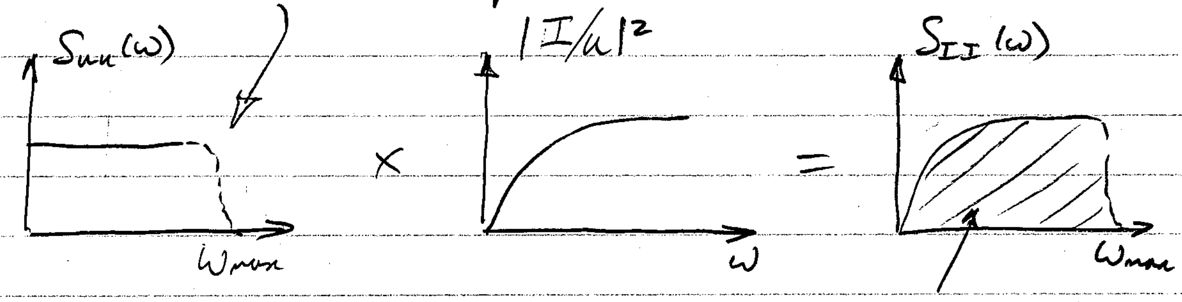


The bigger ω_{max}, the bigger the integral

$$2\int_0^{\omega_{max}} \frac{(\omega/\omega_0)^2}{1+(\omega/\omega_0)^2}\, d\omega$$

So $\bar{P}$ is infinite, which cannot be achieved in practice

The input spectral density $S_{uu}(\omega)$ is inevitably band limited in practice



Integral is finite

This means that the power will be finite