EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 24 April 2024 2.00 to 3.40

Module 4C7

RANDOM AND NONLINEAR VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed.

Attachment: 4C7 Random and Non-linear Vibrations data sheet (4 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A nonlinear, undamped, single degree-of-freedom vibratory system, of mass m is supported by a spring arrangement as shown in Fig. 1. Springs k_2 engage after the mass displaces a distance d either side of the nominal equilibrium position. The mass is sinusoidally driven at an angular frequency ω and the response amplitude is α.

(a) Sketch the force-displacement characteristic for the system. [10%]

(b) Sketch the input and output waveforms when the system is sinusoidally driven at an angular frequency ω with a response amplitude $\alpha > d$. [20%]

(c) Determine the Describing Function for a sinusoidal input corresponding to a response amplitude $\alpha > d$ and show that this reverts to the case of a simple linear spring when $d = 0$. [50%]

(d) Write an equation to determine the response amplitude α over a range of frequencies ω for $\alpha > d$ when the system is driven by an external force $f\cos\omega t$. [20%]

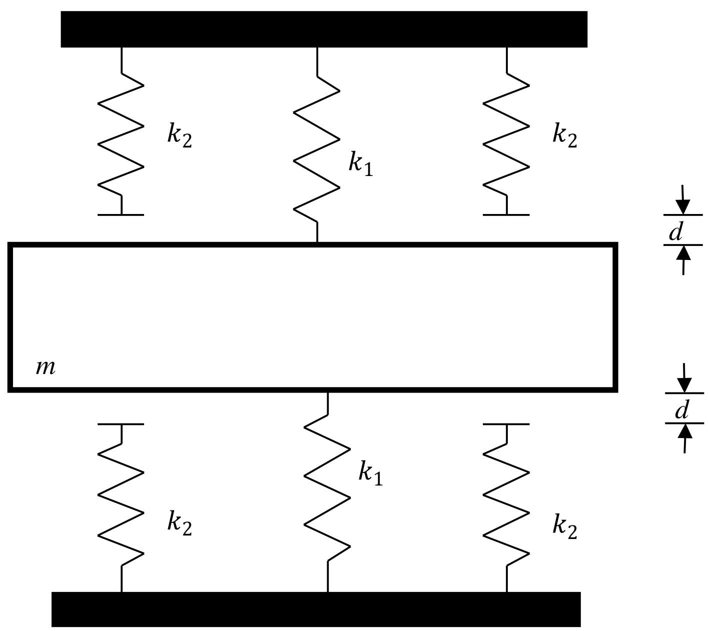

Fig. 1

2 A self-excited nonlinear system is characterised by the equation below, where parameters α, β, γ, δ are all positive constants.

$$\ddot{x} - (\beta - \delta\dot{x}^2)\dot{x} - \alpha x + \gamma x^3 = 0$$

(a) Determine the equilibrium points of the system and their type and stability. [30%]

(b) Sketch the behaviour of the system in the phase plane for $\beta^2 < 8\alpha$. [30%]

(c) Obtain an approximate expression for the amplitude of limit cycle oscillations about the origin by assuming a solution of the form $x = A\cos\Omega t$, where A is the amplitude of the limit cycle and Ω is the oscillation frequency. [40%]

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3 A structural member in part of an offshore platform is stressed by pounding waves and the stress $s(t)$ can be approximated by a narrow-band random process with mean–square spectral density $S_s(\omega)$ as shown in Fig. 2.

(a) Calculate the mean stress and the standard deviation of the stress, and estimate the standard deviation of $\dot{s} = ds/dt$. [10%]

(b) Assuming that s and $\dot{s}$ are independent Gaussian random processes, calculate the frequency of positive–slope crossing of the level $s = a$. You may assume:

$$v_a^+ = \int_0^\infty \dot{s} p(a, \dot{s}) \ d\dot{s}$$

where $p(a, \dot{s})$ is the joint probability distribution. [20%]

(c) Show that the probability distribution of the peaks in stress is

$$p_p(s) = \frac{s}{2S_0\Delta\omega} exp\left(-\frac{s^2}{4S_0\Delta\omega}\right).$$ [20%]

(d) What is the expected number of cycles with peaks in the range s to $s + ds$ during time T ? [20%]

(e) If the number of cycles to failure N_f at stress level s is given by $N_f(s) = Cs^{-k}$, where C and k are constants, determine an integral expression for the expected life of the member and evaluate it for $k = 1$. You may assume that Miner's Rule applies, i.e. failure occurs when $\sum \frac{n(s)}{N_f(s)} = 1$, where $n(s)$ is the number of stress cycles at level s. [30%]

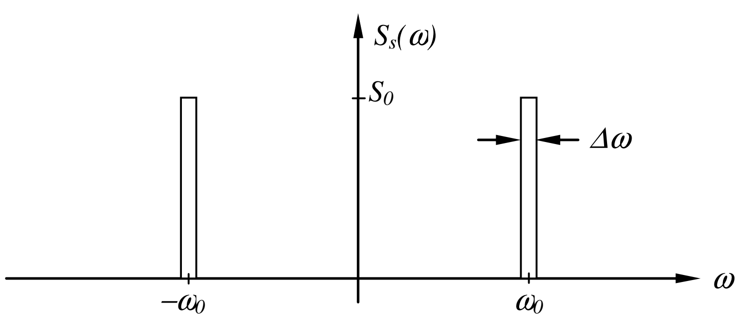

Fig. 2

4 The electric circuit shown in Fig. 3 has source voltage $u(t)$, resistance R and capacitance C. The output voltage measured across the capacitor is $v(t)$ and the current in the resistor is $i(t)$.

(a) If the input voltage $u(t)$ is sinusoidal with amplitude U at angular frequency ω, and the output voltage $v(t)$ has amplitude V, derive an expression for the transfer function $V(j\omega)/U(j\omega)$, where $j = \sqrt{-1}$. [20%]

(b) $u(t)$ is a zero-mean, white noise, random signal with constant mean square spectral density $S_{uu}(\omega) = S_0,\ -\infty < \omega < \infty$.

(i) Write an expression for the mean square spectral density of the output voltage $S_{vv}(\omega)$ and sketch its form. [10%]

(ii) Calculate the mean-square output voltage σ_v^2. [10%]

(iii) Verify that with a suitable value for the constant ω_0, the autocorrelation function of the output can be written in the form: $R_{vv}(\tau) = \pi S_0 \omega_0 e^{-\omega_0 |\tau|}$ [20%]

(c) Derive an integral expression for the mean power dissipated in resistor R. [20%]

(d) With the aid of sketches, explain why it is not possible in practice for $u(t)$ to be white noise at all frequencies. [20%]

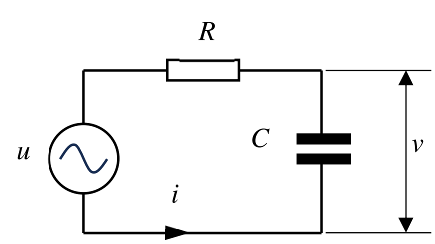

Fig. 3

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PART IIB Module 4C7
Random and Non-linear Vibrations Data Sheet

Part One: Random Vibration

Gaussian Probability Distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{\sigma_y^2(x-\mu_x)^2 + \sigma_x^2(y-\mu_y)^2 - 2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)}{2\sigma_x^2\sigma_y^2(1-\rho^2)} \right\}$$

$$\int_0^\infty e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}, \qquad \int_0^\infty x e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^2, \qquad \int_0^\infty x^2 e^{-x^2/(2\sigma^2)}\mathrm{d}x = \sigma^3\sqrt{\frac{\pi}{2}}$$

Crossing rates: general case

$$\nu_b^+ = \int_0^\infty \dot{x} p(b,\dot{x})\mathrm{d}\dot{x}$$

Crossing rates: Gaussian case

$$\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp\left\{ -\frac{b^2}{2\sigma^2} \right\}$$

Peak distribution for a narrow band process: general case

$$P(b) = 1 - \frac{\nu_b^+}{\nu_0^+}, \qquad p(b) = -\frac{\mathrm{d}}{\mathrm{d}b}\left(\frac{\nu_b^+}{\nu_0^+}\right)$$

Peak distribution for a narrow band process: Gaussian case

$$p(b) = \frac{b}{\sigma_x^2}\exp\left\{ -\frac{b^2}{2\sigma_x^2} \right\}$$

Probability of failure after duration T

$$P_f = 1 - \exp\left\{ -\nu_b^+ T \right\}$$

Spectral relations

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}\mathrm{d}\tau, \qquad R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\omega\tau}\mathrm{d}\omega$$

Input-output relations

$$S_{xx}(\omega) = |H(\omega)|^2 S_{FF}(\omega)$$

$$S_{x_i x_j}(\omega) = \sum_r \sum_s H_{ir}^*(\omega)H_{js}(\omega)S_{F_r F_s}(\omega)$$

Calculation of mean-square response integrals

$$I_n = \int_{-\infty}^{\infty} |H_n(i\omega)|^2 d\omega$$

where $H_n(i\omega) = \dfrac{B_0 + (i\omega)B_1 + ... + (i\omega)^{n-1}B_{n-1}}{A_0 + (i\omega)A_1 + ... + (i\omega)^n A_n}$,

and the roots λ of the characteristic equation $\left(A_0 + \lambda A_1 + \lambda^2 A_2 + ... + \lambda^n A_n = 0\right)$ all have negative real parts, takes the values:

$$I_1 = \pi\frac{B_0^2}{A_0 A_1} \qquad I_2 = \pi\frac{A_0 B_1^2 + A_2 B_0^2}{A_0 A_1 A_2}$$

$$I_3 = \pi\frac{A_0 A_1 B_2^2 + A_2 A_3 B_0^2 + A_0 A_3\left(B_1^2 - 2B_0 B_2\right)}{A_0 A_3 (A_1 A_2 - A_0 A_3)}.$$

White noise input: standard form

$m\ddot{x} + c\dot{x} + kx = G, \qquad S_{GG}(\omega) = S_0$ (double sided)

$$\sigma_x^2 = \frac{\pi S_0}{ck}, \qquad \sigma_{\dot{x}}^2 = \frac{\pi S_0}{cm}$$

White noise input: scaled form

$\ddot{x} + 2\beta\omega_n\dot{x} + \omega_n^2 x = F, \qquad S_{FF}(\omega) = S_0$ (double sided)

$$\sigma_x^2 = \frac{\pi S_0}{2\beta\omega_n^3}, \qquad \sigma_{\dot{x}}^2 = \frac{\pi S_0}{2\beta\omega_n}$$

Fatigue Damage

$\mathrm{E}[D] = \mathrm{E}[1/N(S)]\nu_0^+ T$ where S is proportional to the peak height b

End of Part One

Part Two: Nonlinear Vibration

Describing functions

System described by the undamped Duffing equation subject to harmonic forcing:

$$\ddot{x} + p^2 x + \mu x^3 = a\cos\omega t$$

$$x \approx \alpha\cos\omega t$$

$$\text{Describing Function} = \frac{output}{input} = \frac{3\mu\alpha^2}{4}$$

Classification of equilibrium or singular points

(nonlinear system)

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

(linearised about singular point (0,0))

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Eigenvalues of matrix A	**Type of singular point**
Both real and positive	Unstable node
Both real and negative	Stable node
Both real but opposite sign	Saddle point
Both complex with real part = 0	Centre
Both complex with real part > 0	Unstable focus
Both complex with real part < 0	Stable focus

T=tr(**A**) and *D*=det(**A**)

Conditions on *T* and *D*	**Type of singular point**
$D < 0$	Saddle point
$D > 0, T = 0$	Centre
$D > 0, -2\sqrt{D} < T < 0$	Stable focus
$D > 0, 0 < T < 2\sqrt{D}$	Unstable focus
$D > 0, T > 2\sqrt{D}$	Unstable node
$D > 0,\ T < -2\sqrt{D}$	Stable node

Classification of equilibrium or singular points for conservative systems:

(nonlinear system) $$\frac{1}{2}m\dot{x}^2 + V(x) = E$$

Sign of $V''(x)$	**Type of singular point**
positive	Centre
negative	Saddle point

End of Part Two

RSL/AAS/DC
December 2020