Prof D Cebon

Part IIB Module 4C7 2014
Solutions

Q 2 (b) $R_f(\tau) = A\, e^{-b|\tau|}$

Using the Fourier Transform relationship between $R(\tau)$ and $S(\omega)$:

$$S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A e^{-b|\tau|} e^{-i\omega\tau}\, d\tau$$

$$= \frac{A}{2\pi}\int_{-\infty}^{0} e^{(b-i\omega)\tau}\, d\tau + \frac{A}{2\pi}\int_{0}^{\infty} e^{-(b+i\omega)\tau}\, d\tau$$

$$= \frac{A}{2\pi}\frac{(1-0)}{(b-i\omega)} - \frac{A(0-1)}{2\pi(b+i\omega)} = \frac{Ab/\pi}{b^2+\omega^2}$$

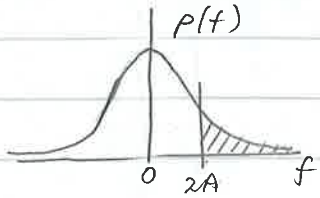
$$= \underline{\underline{\frac{A/b\pi}{1+(\omega/b)^2}}} \qquad \underline{\underline{\text{So}}} \quad S_0 = \frac{A}{b\pi} \quad \& \quad \omega_0 = b$$

(a) The mean square value of f is $R_x(0) = A$

The mean value of f is zero & $\sigma_x = \sqrt{A}$

So the probability density function for f is a Gaussian:

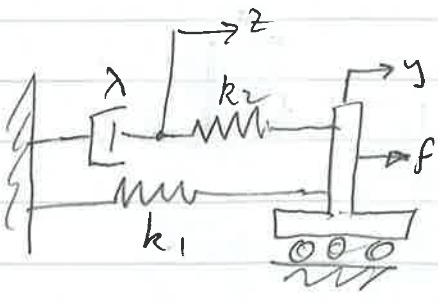
$$\text{Prob}(f > 2\sqrt{A}) = \int_{2\sqrt{A}}^{\infty} \frac{1}{\sqrt{2\pi A}} e^{-f^2/2A}\, df$$



$z = f/\sigma = f/\sqrt{A}$

$$\Rightarrow \Pr(z>2) = \int_{2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \underline{\underline{0{\cdot}023}} \quad (\text{tables})$$

(c)



$$k_1 y + k_2(y-z) = f \quad \text{—— (1)}$$

$$k_2(y-z) - \lambda\dot{z} = 0 \quad \text{—— (2)}$$

Let $f = Fe^{i\omega t}$, $y = Ye^{i\omega t}$ & $z = Ze^{i\omega t}$

2(c) cont

$$\textcircled{2} \Rightarrow \quad Z(i\omega\lambda + k_2) = k_2 Y \Rightarrow \frac{Z}{Y} = \frac{k_2}{i\omega\lambda + k_2} \quad \text{—} \textcircled{3}$$

$$\textcircled{3} \text{ into } \textcircled{1} \Rightarrow (k_1 + k_2)Y - \frac{k_2^2 Y}{i\omega\lambda + k_2} = F$$

$$\Rightarrow \frac{Y}{F} = \frac{k_2 + i\omega\lambda}{k_1 k_2 + (k_1 + k_2) i\omega\lambda} \quad \text{—} \textcircled{4}$$

$$E[y^2] = \int_{-\infty}^{\infty} S_y(\omega)\, d\omega = \int_{-\infty}^{\infty} |H(\omega)|^2 S_f(\omega)\, d\omega$$

$$= S_0 \int_{-\infty}^{\infty} \left(\frac{k_2 + i\omega\lambda}{k_1 k_2 + (k_1+k_2) i\omega\lambda}\right)^2 \left(\frac{S_0}{1 + (\omega/\omega_0)^2}\right) d\omega$$

$$= S_0 \int_{-\infty}^{\infty} \left| \frac{k_2 + i\omega\lambda}{(1 + i\omega/\omega_0)(k_1 k_2 + (k_1 + k_2) i\omega\lambda)} \right|^2 d\omega$$

This is a standard integral of the form

$$I = \int_{-\infty}^{\infty} \left| \frac{B_0 + i\omega B_1}{A_0 + (i\omega)A_1 + (i\omega)^2 A_2} \right|^2 d\omega$$

with $B_0 = k_2$, $B_1 = \lambda$, $A_0 = k_1 k_2$, $A_1 = \frac{k_1 k_2}{\omega_0} + (k_1 + k_2)\lambda$

& $A_2 = (k_1 + k_2)\lambda / \omega_0$

for which $I = \pi \dfrac{\{A_0 B_1^2 + A_2 B_0^2\}}{A_0 A_1 A_2}$ (see data sheet)

Giving $E[y^2] = \dfrac{S_0 \pi \left[k_1 \cancel{k_2} \lambda^{\cancel{2}} + (k_1 + k_2)\cancel{\lambda} k_2^{\cancel{2}}/\omega_0\right]}{k_1 \cancel{k_2} \left[\frac{k_1 k_2}{\omega_0} + (k_1 + k_2)\lambda\right]\left[\frac{(k_1 + k_2)\cancel{\lambda}}{\omega_0}\right]}$

$$= \frac{S_0 \pi \omega_0}{k_1} \left\{ \frac{\omega_0 k_1 \lambda + k_1 k_2 + k_2^2}{[k_1 k_2 + \omega_0 (k_1 + k_2)\lambda][k_1 + k_2]} \right\} \quad //$$

with $S_0 = \dfrac{A}{b\pi}$ & $\omega_0 = b$

1 (a) As there is no delta function in $S_s(\omega)$ at $\omega = 0$, $M_s = 0$

Hence $\sigma_s^2 = E[s^2] = \int_{-\infty}^{\infty} S_s(\omega)\, d\omega = \underline{\underline{2 S_0 \Delta\omega}}$

$$\sigma_{\dot{s}}^2 = \int_{-\infty}^{\infty} \omega^2 S_s(\omega)\, d\omega = \underline{\underline{2 S_0 \omega_0^2 \Delta\omega}}$$

(b) Assuming s & $\dot{s}$ are independent Gaussian processes

$$p(s, \dot{s}) = \frac{1}{2\pi \sigma_s \sigma_{\dot{s}}} e^{-s^2/2\sigma_s^2} e^{-\dot{s}^2/2\sigma_{\dot{s}}^2}$$

$$\therefore \nu_a^+ = \int_0^{\infty} \frac{e^{-a^2/2\sigma_s^2}}{2\pi \sigma_s \sigma_{\dot{s}}} \dot{s} e^{-\dot{s}^2/2\sigma_{\dot{s}}^2} d\dot{s} = \frac{e^{-a^2/2\sigma_s^2}}{2\pi \sigma_s \sigma_{\dot{s}}} \underbrace{\int_0^{\infty} \dot{s} e^{-\dot{s}^2/2\sigma_{\dot{s}}^2} d\dot{s}}_{\substack{\sigma_{\dot{s}}^2 \\ \text{(data sheet)}}}$$

$$= \frac{e^{-a^2/2\sigma_s^2}}{2\pi \sigma_s \sigma_{\dot{s}}} \sigma_{\dot{s}}^2 = \frac{1}{2\pi} \frac{\sigma_{\dot{s}}}{\sigma_s} e^{-a^2/2\sigma_s^2}$$

Substituting for σ_s & $\sigma_{\dot{s}}$ from (a) gives

$$\underline{\underline{\nu_a^+ = \frac{1}{2\pi} \omega_0 e^{-a^2/4S_0\Delta\omega}}}$$

(c) Prob (peak $> a$) = proportion of peaks $> a$

$$= \int_a^{\infty} p_p(s)\, ds = \frac{\nu_a^+}{\nu_0^+} = e^{-a^2/4S_0\Delta\omega}$$

$$\therefore p_p(s) = -\frac{d}{da}\left(e^{-a^2/4S_0\Delta\omega}\right)\Bigg|_{a=s} = \underline{\underline{\frac{s}{2S_0\Delta\omega} e^{-s^2/4S_0\Delta\omega}}}$$

(d) Expected number of cycles with peaks in the range s to $s+ds$ in time T

$= (\text{Total number of cycles in } T) \times \text{Prob}\,(s < \text{peak} < s+ds)$

$= \nu_0^+ T \times p_p(s)\, ds = \underline{\underline{\frac{s\,\omega_0 T}{4\pi S_0 \Delta\omega} e^{-s^2/4S_0\Delta\omega} ds}}$

1(e) Number of cycles to failure at stress level s is $N_f(s) = C s^{-k}$

Total number of cycles at this level in time T is $\nu_0^+ T p_p(s)\, ds$

$\therefore$ failure occurs (according to Miner's rule) when:

$$\int_0^\infty \frac{\nu_0^+ T p_p(s)\, ds}{N_f(s)} = 1$$

$$\Rightarrow \int_0^\infty \frac{\omega_0 T}{4\pi S_0 \Delta\omega C} s^{k+1} e^{-s^2/4S_0\Delta\omega}\, ds = 1$$

$$\therefore T = \underline{\underline{\left[\int_0^\infty \frac{\omega_0}{4\pi S_0 \Delta\omega C} s^{k+1} e^{-s^2/4S_0\Delta\omega}\, ds \right]^{-1}}}$$

for $k=1$ $$T = \left[\frac{\omega_0}{4\pi S_0 \Delta\omega C} \underbrace{\int_0^\infty s^2 e^{-s^2/2\sigma_s^2}\, ds}_{\sigma_s^3\sqrt{\pi/2}\ \text{(data sheet)}} \right]^{-1}$$

So $$T = \left[\frac{\omega_0}{4\pi S_0 \Delta\omega C} \sqrt{\frac{\pi}{2}} (2S_0\Delta\omega)^{3/2} \right]^{-1}$$

$$= \underline{\underline{\frac{2\sqrt{\pi}\, C}{\omega_0 \sqrt{S_0 \Delta\omega}}}}$$

Q3 (a) $m\ddot{x} = -k\left[\sqrt{x^2+l^2} - \hat{l}\right]\cdot\dfrac{x}{\sqrt{x^2+l^2}}$

$$m\ddot{x} + \frac{kx}{\sqrt{x^2+l^2}}\left[\sqrt{x^2+l^2} - \hat{l}\right] = 0$$

$\hat{l}$: undeformed spring length

(b) $\dfrac{1}{l}\dfrac{\ddot{x}}{\left(\frac{k}{2m}\right)} + \dfrac{2x/l}{\sqrt{1+\left(\frac{x}{l}\right)^2}}\left[\sqrt{1+\left(\frac{x}{l}\right)^2} - \dfrac{\hat{l}}{l}\right] = 0.$

substituting $u = x/l$, $\tau = \omega t$ where $\omega^2 = \dfrac{k}{2m}$,

$\lambda = \dfrac{\hat{l}}{l}$ we get:

$$\ddot{u} + 2u(1+u^2)^{-1/2}\left[(u^2+1)^{1/2} - \lambda\right] = 0$$

for free oscillation

(c) $\ddot{u} + 2u\left[1 - \frac{1}{2}u^2\right]\left[1 + \frac{1}{2}u^2 - 1\right] \simeq 0$, u small

$$\ddot{u} + 2u\left[\frac{1}{2}u^2 - \frac{1}{4}u^4\right] \simeq 0$$

or $\ddot{u} + u^3 = 0$, u small

retaining first higher order term for free oscillation

(d) Harmonic Balance:

$$u = A\cos t + B\cos 3t$$
$$\ddot{u} = -A\cos t - 9B\cos 3t$$

$$-A\cos t - 9B\cos 3t + (A\cos t + B\cos 3t)^3 \simeq a\cos t$$

$$-A\cos t - 9B\cos 3t + A^3\cos^3 t + B^3\cos^3 3t$$
$$+ 3A^2B\cos^2 t\cos 3t$$
$$+ 3AB^2\cos t\cos^2 3t \simeq a\cos t$$

$$-A\cos t - 9B\cos 3t + A^3\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right)$$
$$+ B^3\left(\frac{3}{4}\cos 3t + \frac{1}{4}\cos 9t\right) + 3A^2B\left(\frac{1+\cos 2t}{2}\right)\cos 3t$$
$$+ 3AB^2\left(\frac{1+\cos 6t}{2}\right)\cos t \simeq a\cos t$$

Retaining terms upto $\cos t$, $\cos 3t$ only:

$$-A\cos t - 9B\cos 3t + \frac{3}{4}A^3\cos t + \frac{A^3}{4}\cos 3t$$
$$+ \frac{3}{4}B^3\cos 3t + \frac{3A^2B}{2}\cos 3t + \frac{3A^2B}{4}\cos t$$
$$+ \frac{3AB^2}{2}\cos t \simeq a\cos t.$$

Equating terms in $\cos t$:

$$-A + \frac{3}{4}A^3 + \frac{3A^2B}{4} + \frac{3AB^2}{2} \simeq a$$

$$-9B + \frac{A^3}{4} + \frac{3B^3}{4} + \frac{3A^2B}{2} = 0.$$

These equations are to be solved simultaneously to obtain values for A, B.

Q4 (a)

$$\dot{x} = y$$
$$\dot{y} = \mu y + x - x^2 + xy$$

Equilibrium points when:

$$\dot{x} = \dot{y} = 0$$
$$y = 0$$

and $x - x^2 = 0$

$y = 0$, $x = 0, 1$. are the two equilibrium points

(b) Linearising about $(0, 0)$

$$\dot{x} = y$$
$$\dot{y} = \mu y + x$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & \mu \end{bmatrix}.$$

eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & \mu-\lambda \end{vmatrix} = 0$$

$$-\lambda(\mu-\lambda) - 1 = 0$$

$$\lambda^2 - \mu\lambda - 1 = 0$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2+4}}{2}$$

$\mu < 0$ $\therefore$ equilibrum point is a saddle point

Take equilibrum point $(1,0)$ & linearize about $(0,0)$

define $z = x - 1$

$$\dot{z} = y$$
$$\dot{y} = \mu y + (z+1)(-z) + (z+1)y$$
$$= \mu y - z^2 - z + zy + y.$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu+1 \end{bmatrix}$$

eigenvalues given by:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \mu+1-\lambda \end{vmatrix} = 0$$

$$-\lambda(\mu+1-\lambda) + 1 = 0$$

$$\lambda^2 - \lambda(\mu+1) + 1 = 0$$

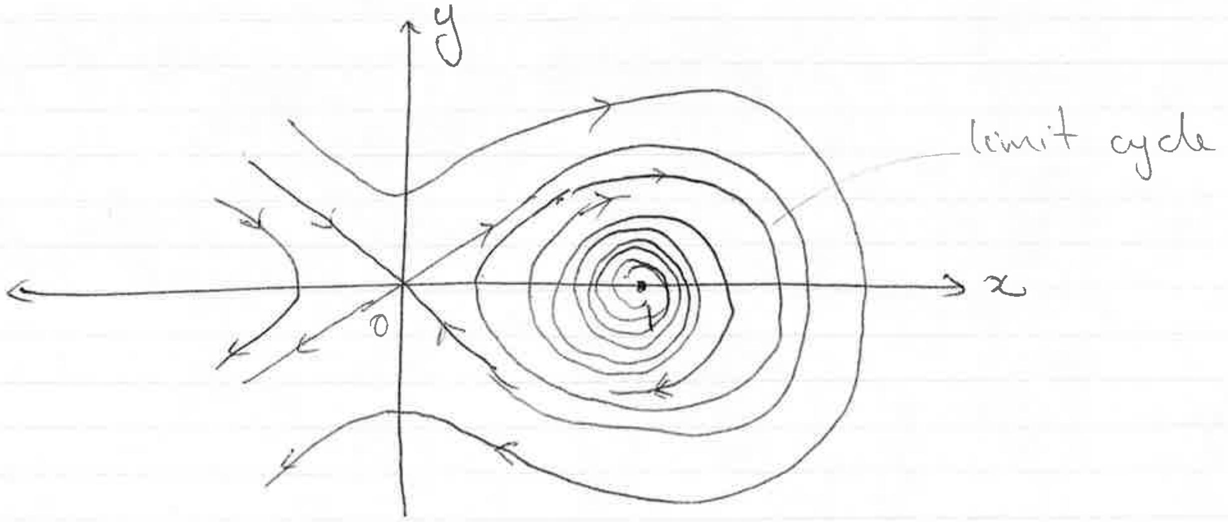
$$\lambda = \frac{(\mu+1) \pm \sqrt{\mu^2+2\mu+1-4}}{2}$$

$$\lambda = \frac{(\mu+1) \pm \sqrt{(\mu+3)(\mu-1)}}{2}$$

$\mu < 0$, $\mu > -3 \Rightarrow$ complex conjugate pair with either +ve or -ve real part depending on whether $\mu < -1$ or $\mu > -1$

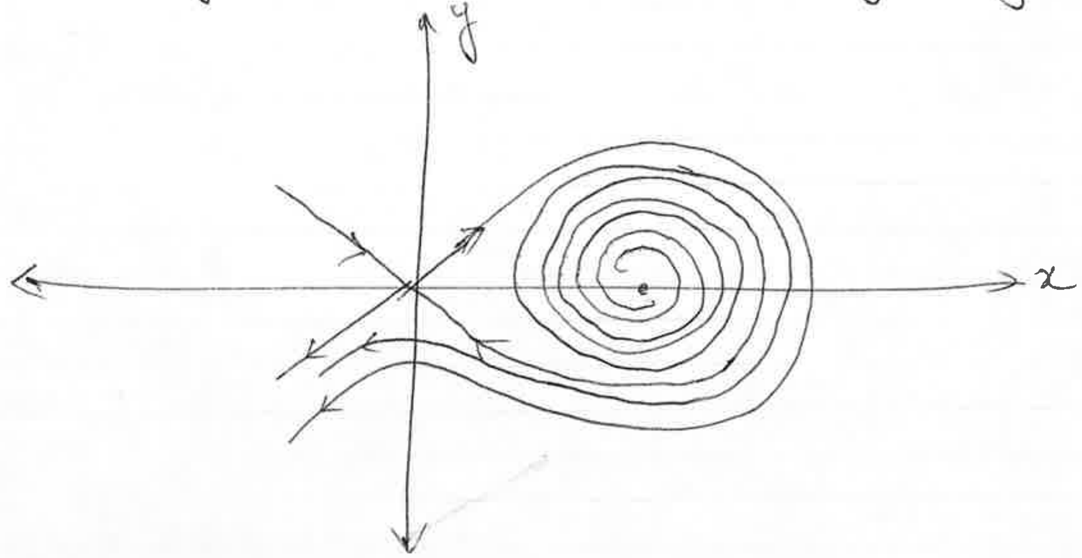
$\therefore$ $-1 < \mu < 0$, unstable focus
$\mu = -1$, center
$-3 < \mu < -1$, stable focus
$\mu < -3$, roots real & -ve $\therefore$ stable node.

(c)



The limit cycle may be "fully enclosed" by the trajectorys passing through the saddle point, coincident with (for a critical value of μ) or "break" the saddle connection depending on the value of μ.

"Breaking" the saddle point trajectory for $\mu > \mu_c$



This is an example of a homoclinic bifurcation