4C7 Cribs 2022

1. a) $S_{zz}(\omega)$ is the Fourier transform of $R_{zz}(\tau)$. Now:

$$R_{zz}(\tau) = E[z(t)\,z(t+\tau)]$$

$$= E[\{f(t)+\alpha a(t)\}\{f(t+\tau)+\alpha a(t+\tau)\}]$$

$$= E[f(t)f(t-\tau) + \alpha^2 a(t)a(t+\tau) + \underbrace{\alpha a(t) f(t+\tau) + \alpha a(t+\tau) f(t)}]$$

$$= R_{ff}(\tau) + \alpha^2 R_{aa}(\tau)$$

each term has zero average since $a(t)$ and $f(t)$ are independent

$$\Rightarrow \underline{S_{zz}(\omega) = S_{zz}(\omega) + \alpha^2 S_{aa}(\omega)}$$

[10%]

b) $M\ddot{x} + c(\dot{x}-\dot{u}) + k(x-u) = F$ where u = ground displacement

Put $r = x - u \Rightarrow M\ddot{r} + c\dot{r} + kr = F - M\ddot{u} = F - Ma$

from (a), spectrum $= S_{FF} + M^2 S_{aa}$

White noise result $\sigma_r^2 = \dfrac{\pi}{2cM}\left[S_{FF}(\omega_n) + M^2 S_{aa}(\omega_n)\right]$

note white noise approximation, evaluated at $\omega = \omega_n$

$$\underline{\sigma^2 = \frac{\pi}{2c}\left[\frac{S_{FF}}{M} + M S_a\right]}$$

[60%]

c) Follows from above that $P = c\,\sigma_r^2 = \dfrac{\pi}{2}\left[\dfrac{S_{FF}}{M} + M S_a\right]$

[20%]

d) $M\ddot{x} + c(\dot{x} - \dot{u}) + k(x-u) = F$

$M\ddot{x} + c\dot{} + kx = F + c\dot{u} + ku$

$\downarrow$

Frequency domain $\left(\frac{c}{i\omega} \quad \frac{k}{\omega^2}\right) a$

Spectrum $\left|\frac{c}{i\omega} \quad \frac{k}{\omega^2}\right|^2 S_{aa}(\omega) = \left(\frac{c^2}{\omega} + \frac{k^2}{\omega^4}\right) S_{aa}(\omega)$

$\Rightarrow \quad \sigma_x^2 = \frac{\pi}{2ck}\left[S_{FF}(\omega_n) + \left(\frac{c^2}{\,_2} + \frac{k^2}{\omega_n^4}\right) S_{aa}(\omega_n)\right] \qquad \omega^2 = k/M$

$$\underline{\sigma_x^2 \quad \frac{\pi}{2c}\left[\frac{S_{FF}(\omega_n)}{k} + \left(\frac{Mc^2}{k} + \frac{M^2}{k}\right) S_{aa}(\omega_n)\right]}$$

Appears that $\sigma_x \to 0$ as $k \to \infty$

<u>But</u> this assumes white noise so response is <u>always resonant</u>

<u>In reality</u> for $k \to \infty$ $S_{aa}(\omega)$ will be zero (not ideal white noise) for a frequency ω with $\omega \ll \omega_n$. So response will be stiffness dominated, not resonant, and $\sigma_x \to \sigma_u$.

[30%]

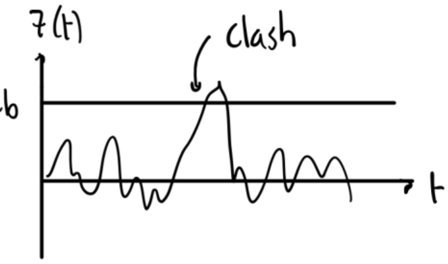
2. (a) $p = 10 = 1 - \cancel{\ln}(p)$; $\nu^+_{(2b)} = \nu^+ e$ (b) p 10

$\overrightarrow{0}$ $\overrightarrow{0}$

$\overleftrightarrow{2b}$

$10^{-3} = 1 - \exp\{-50 \; e^{-\frac{1}{2}(b/\sigma_x)^2}\}$

Put $z(t) = x_1(t) - x_2(t)$



⇒ 103 ₅t up

s~~o~~o é blat

A clash will occur whn $z(t) \geq 2b$

$z^2(t) = x^2 + x_2^2 - 2x_1x_2 \Rightarrow \sigma_z^2 = \sigma_x^2 + \sigma_x^2 - 2\rho\sigma_x^2$

$= \sigma_x^2 \, 2(1-\rho)$

$\dot{z}^2(t) = \dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}\dot{x} \Rightarrow \sigma_{\dot{}}^2 = \sigma^2 \, 2(1-\rho)$

For $z(t)$, $\nu^+_{2b} = \frac{1}{2\pi}(\sigma_{\dot{z}}/\sigma_z)\exp\{-\frac{1}{2}\left(\frac{2b}{\sigma_z}\right)^2\}$

$\nu^+_{2b} = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp\{-\frac{1}{2}\left(\frac{2b^2}{\sigma_x^2(1-\rho)}\right)\}$

$\underline{p = 1 - e^{-\nu^+_{2b}T}}$

Maximum when $\rho = -1 \Rightarrow \nu^+_{2b} = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp\{-\frac{1}{2}\left(\frac{b}{\sigma_x}\right)^2\} = (\nu^+_b)_{\text{for}}$

Obviously tu s m for $\rho = -1$, $x_2(t) = -x(t)$, and ther is a clash

whn $x(t) = b$ [35%]

[35%]

3 2 σ

$\Rightarrow -5000\, e^{-\frac{1}{2}(b/\sigma_x)^2} = \ln(1-10^{-3})$

$\Rightarrow \frac{1}{2}(b/\sigma_x)^2 = -\ln\left[\frac{-\ln(1-10^{-3})}{5000}\right] \Rightarrow \underline{b/\sigma = 5.55}$

[30%]

(c) Fo a single random stress $x(t)$

$$E[D] = \nu_0^+ T \; E[1/N(s)] \quad \text{where } N(s) \text{ is the S N law}$$

↑ evaluated using peak distributi n

$$E[D] = F(\sigma_x \;\; \sigma_{\dot{x}})$$

In this case for board 1 : $E[D_1] = F(\sigma_{s_1}, \sigma_{\dot{s}_1})$

For board 2 : $E[D_2] = F(\sigma_2, \sigma_{\dot{2}})$

Failure will occur when $\max\{E[D_1], E[D_2]\} = 1 \Rightarrow$ Fat gue life

ie/ compute the Fatigu life of each board and <u>tak the Minimum</u>

(a) Conservative system of unit mass

Potential energy $V(x)$ such that

$$\frac{dV}{dx} = e^{x} - 1$$

$$\therefore V(x) = e^{x} - x + c$$

(b) For origin: $x = 0$, $\ddot{x} = 0$

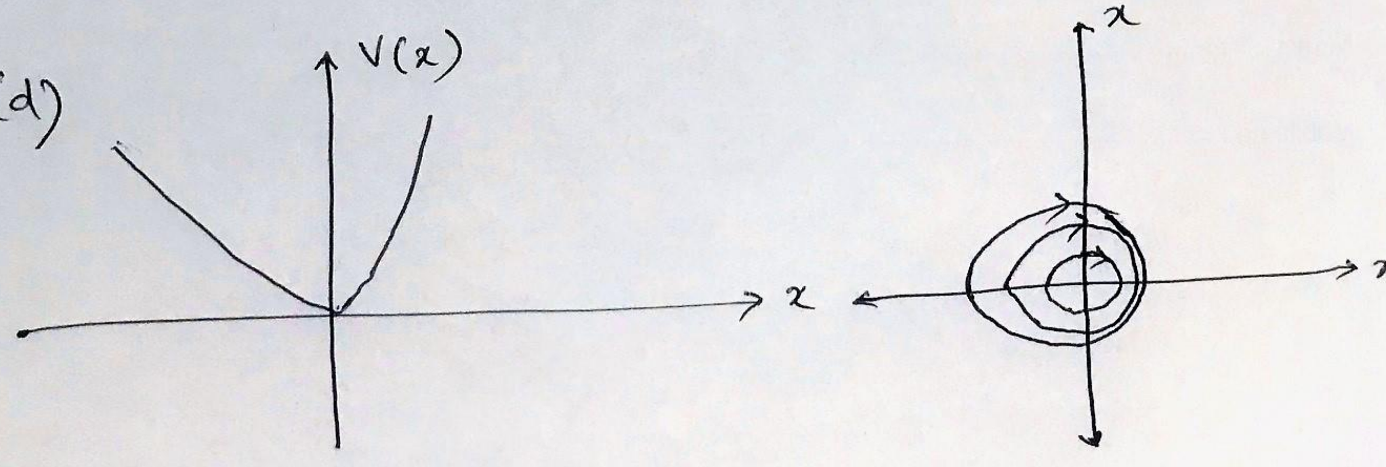
$\therefore$ singular point

Note also $\frac{dV}{dx} = 0$

(c) $$\frac{d^2V}{dx^2} = 1 > 0$$

$\therefore$ we have a centre.

(d)



(e) Equation for phase trajectories for conservative system:

$$\frac{1}{2}\dot{x}^2 + e^{x} - x = E', \text{ where}$$

$E' =$ constant.

(a) Assume $x = A\cos\Omega t$

Substitute into $\ddot{x} + (\alpha - \beta x^2 + \gamma x^4)\dot{x} + x + \mu x^3 = 0$.

$\ddot{x} = -\Omega^2 A\cos\Omega t$, $\dot{x} = -A\Omega\sin\Omega t$.

Truncating to $\cos\Omega t$, $\sin\Omega t$ terms we get:

$$\left(-\Omega^2 A + A + \frac{3}{4}\mu A^3\right)\cos\Omega t + \left(\alpha - \frac{\beta A^2}{2} + \frac{\beta A^2}{4} + \frac{3\gamma A^4}{8} - \frac{\gamma A^4}{4}\right) \times -A\Omega\sin\Omega t = 0.$$

Equating terms in $\cos\Omega t$ and $\sin\Omega t$ on LHS and RHS and assuming $A \neq 0$.

$$-\Omega^2 + 1 + \frac{3}{4}\mu A^2 = 0$$

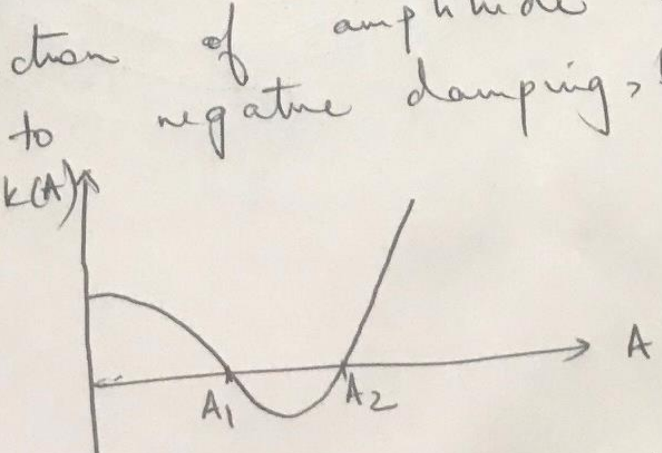
$$\text{or} \quad \Omega^2 = 1 + \tfrac{3}{4}\mu A^2. \qquad \text{—(1)}$$

$$K(A) = \alpha - \frac{1}{4}\beta A^2 + \frac{1}{8}\gamma A^4 = 0.$$

$$\Rightarrow A = \left[\frac{\beta}{\gamma} \pm \sqrt{\left[\left(\frac{\beta}{\gamma}\right)^2 - \frac{8\alpha}{\gamma}\right]}\,\right]^{1/2}$$

Two roots exist corresponding to the amplitudes of the two limit cycles.

(b) For $\beta^2 > 8\alpha\gamma$ $K(A)$ – the damping as a function of amplitude transitions from positive to negative damping, back to positive



... cycles exist, one stable and the

(c) The origin is a singular point

$\dot{x} = 0$, $\ddot{x} = 0$. at the origin.

and can be shown to be stable