Note for Q1: Take care to distinguish subscript ms (mean square) from m_S (sprung mass)

1 a) i) $k = 0$.

$$A_{ms} = \frac{\pi S_0 \left[k_t c^2 \right]}{m_s^2 c} = \frac{\pi S_0 k_t c}{m_s^2} \quad \text{— (1)}$$

$$D_{ms} = \frac{\pi S_0 (m_s + m_u)}{c} \Rightarrow c = \frac{\pi S_0 (m_s + m_u)}{D_{ms}}$$

subst for c in (1)

$$A_{ms} = \frac{\pi S_0 k_t}{m_s^2} \frac{\pi S_0 (m_s + m_u)}{D_{ms}}$$

$$A_{ms} . D_{ms} = \left(\frac{\pi S_0}{m_s} \right)^2 k_t (m_s + m_u)$$

ii) asymptote as $c \to 0$.

$$A_{ms} \to \frac{\pi S_0 (m_s + m_u) k^2}{m_s^2 c}$$

gradient. $$\frac{A_{ms}}{D_{ms}} \longrightarrow \frac{\frac{\cancel{\pi S_0 (m_s + m_u)} k^2}{m_s^2 \cancel{c}}}{\frac{\cancel{\pi S_0 (m_s + m_u)}}{\cancel{c}}}$$

$$\frac{A_{ms}}{D_{ms}} \to \frac{k^2}{m_s^2}$$

iii) $$A_{ms} = \frac{\pi S_0 \left[(m_s + m_u) k^2 + k_t c^2 \right]}{m_s^2 c}$$

$$\frac{dA_{ms}}{dc} = \frac{\pi S_0}{m_s^2} \left\{ \frac{c(2k_t c) - \left[(m_s + m_u) k^2 + k_t c^2 \right] 1}{c^2} \right\}$$

let $\frac{dA_{ms}}{dc} = 0$ $\qquad k_t c^2 = (m_s + m_u) k^2$

$$c_{opt.} = k \sqrt{\frac{m_s + m_u}{k_t}}$$

b) $$A_{ms} = \frac{\pi S_0 \left[(m_s + m_u) k^2 + k_t c^2 \right]}{m_s^2 c}$$

assumptions lead to $A_{ms} \simeq \left(\frac{\pi S_0}{c} \right) \frac{k^2}{m_s}$

$$D_{ms} = \left(\frac{\pi S_0}{c} \right) (m_s)$$

k proportional to:	A ms proportional to:	D ms proportional to:
const	$\frac{1}{m_s} = m_s^{-1}$	m_s
m_s	$\frac{m_s^2}{m_s} = m_s$	m_s
m_s^2	$\frac{m_s^4}{m_s} = m_s^3$	m_s

c) using $c_{optimum}$

$$A_{ms} = \frac{\pi S_0}{m_s^2} \cdot \frac{(m_s+m_u)k^2 + (m_s+m_u)k^2}{k\sqrt{\frac{m_s+m_u}{k_t}}}$$

$$= \frac{\pi S_0}{m_s^2} 2k\sqrt{(m_s+m_u)k_t}$$

$m_s \gg m_u$ $\quad \therefore A_{ms} \to \left(2\pi S_0 \sqrt{k_t}\right) \cdot \frac{k}{m_s^{3/2}}$ ||

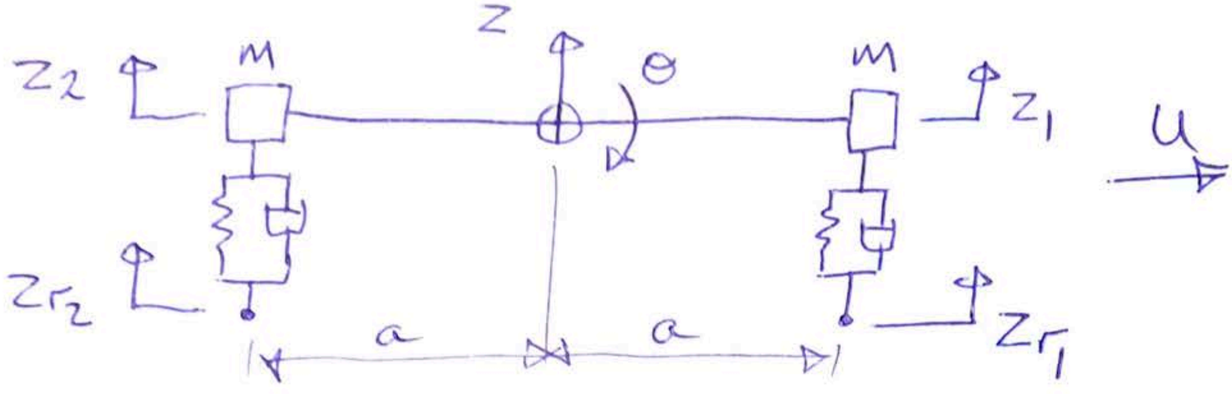
$$D_{ms} = \frac{\pi S_0 (m_s+m_u)\sqrt{k_t}}{k\sqrt{m_s+m_u}}$$

$m_s \gg m_u$ $\quad \therefore D_{ms} \to \left(\pi S_0 \sqrt{k_t}\right) \frac{m_s^{\frac{1}{2}}}{k}$ ||

k proportional to:	A_{ms} proportional to:	D_{ms} proportional to:
const	$1/m_s^{3/2} = m_s^{-\frac{3}{2}}$	$m_s^{\frac{1}{2}}$
m_s	$m_s/m_s^{3/2} = m_s^{-\frac{1}{2}}$	$m_s^{\frac{1}{2}}/m_s = m_s^{-\frac{1}{2}}$
m_s^2	$m_s^2/m_s^{3/2} = m_s^{\frac{1}{2}}$	$m_s^{\frac{1}{2}}/m_s^2 = m_s^{-\frac{3}{2}}$

d)
- Least variation in A_{ms} and D_{ms} with m_s is when $k \propto m_s$ and $c = c_{optimum}$.
- Achieve $k \propto m_s$ using a 'constant volume' air spring. (Use 'constant mass' air spring for $k \propto m_s^2$).
- To achieve $c_{optimum}$ use variable damper using, for example, electrorheological fluid.

2



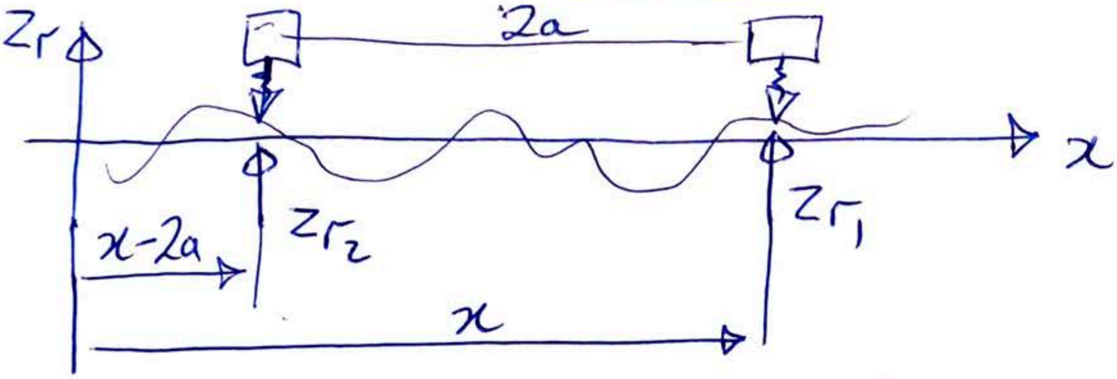
a) note that $I = 2ma^2$ implies mass m above each axle, connected by a light rigid beam

$$z = \frac{z_1 + z_2}{2} \quad , \quad \theta = \frac{z_2 - z_1}{2a} \quad , \quad \frac{z_1(j\omega)}{z_{r_1}(j\omega)} = \frac{z_2(j\omega)}{z_{r_2}(j\omega)} = H(j\omega)$$

data book case (c)

hence
$$\underline{\underline{\begin{Bmatrix} z(j\omega) \\ \theta(j\omega) \end{Bmatrix} = \frac{1}{2} H(j\omega) \begin{bmatrix} 1 & 1 \\ -1/a & 1/a \end{bmatrix} \begin{Bmatrix} z_{r_1}(j\omega) \\ z_{r_2}(j\omega) \end{Bmatrix}}}$$

b)



consider road displ z_r as function of distance x.
then input to front axle is $z_{r_1}(x) = z_r(x)$
rear axle $z_{r_2}(x) = z_r(x - 2a)$

let $t = \dfrac{x}{u}$ $\qquad z_{r_1}(t) = z_r(t)$

$$z_{r_2}(t) = z_{r_1}\left(t - \frac{2a}{u}\right)$$

transform $\qquad z_{r_2}(j\omega) = z_{r_1}(j\omega)\, e^{-j\frac{2a\omega}{u}}$

now $\quad z(t) = \frac{1}{2}\left(z_1(t) + z_2(t)\right)$

$$Z(j\omega) = \frac{1}{2}\left(Z_1(j\omega) + Z_2(j\omega)\right)$$

$$= \frac{1}{2}\left(Z_{r_1}(j\omega) + Z_{r_2}(j\omega)\right) H(j\omega)$$

$$= \frac{1}{2} Z_{r_1}(j\omega)\left(1 + e^{-j\frac{2a\omega}{u}}\right) H(j\omega)$$

$$\frac{Z(j\omega)}{Z_{r_1}(j\omega)} = \frac{1}{2} H(j\omega)\left(1 + \cos\frac{2a\omega}{u} - j\sin\frac{2a\omega}{u}\right)$$

$$\left|\frac{Z(j\omega)}{Z_{r_1}(j\omega)}\right| = \frac{1}{2}\left|H(j\omega)\right|\sqrt{\left(1+\cos\frac{2a\omega}{u}\right)^2 + \sin^2\frac{2a\omega}{u}}$$

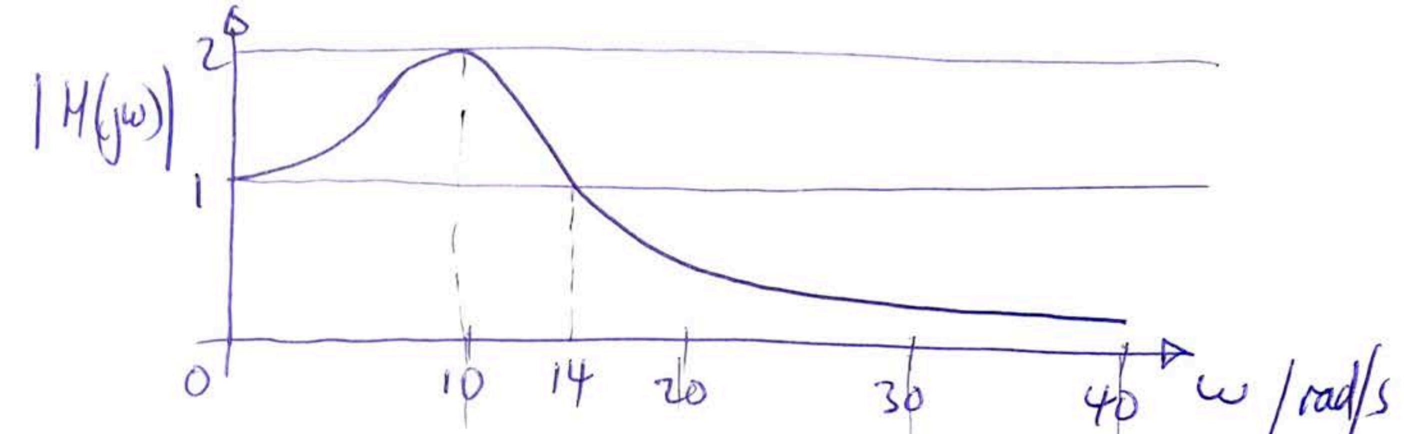
$$= \frac{1}{2}\left|H(j\omega)\right|\sqrt{1 + 2\cos\frac{2a\omega}{u} + \cos^2\frac{2a\omega}{u} + \sin^2\frac{2a\omega}{u}}$$

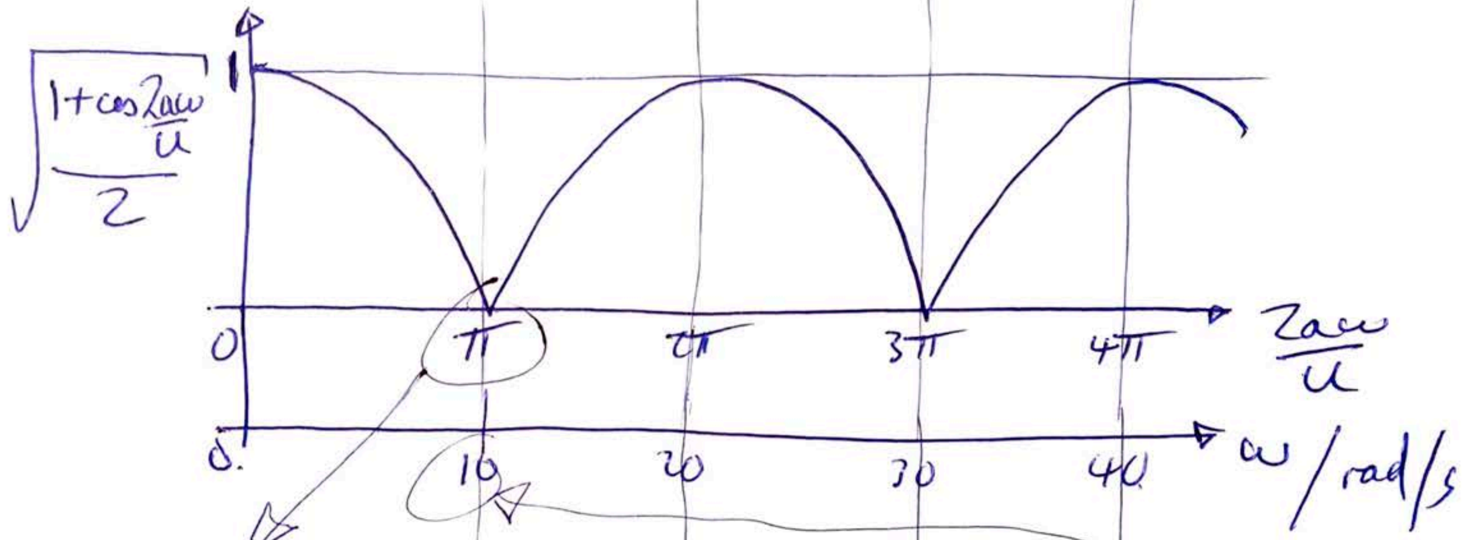
$$= \frac{1}{2}\left|H(j\omega)\right|\sqrt{2 + 2\cos\frac{2a\omega}{u}}$$

$$= \left|H(j\omega)\right|\sqrt{\frac{1 + \cos\frac{2a\omega}{u}}{2}}$$

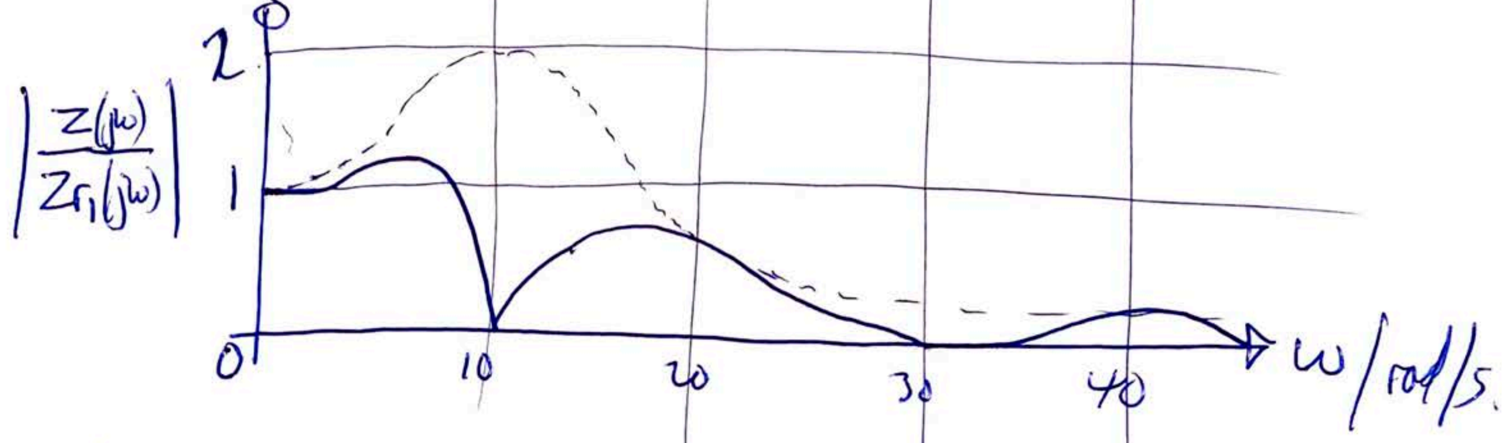
c) $\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{90 \cdot 10^3}{800}} \simeq 10.6 \text{ rad/s}$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{5000}{2\sqrt{90 \cdot 10^3 \cdot 800}} = 0.3$$





$$\frac{2a\omega}{u} = \pi \Rightarrow \omega = \frac{\pi u}{2a} = \frac{\pi \cdot 8}{2 \cdot 1.3} \simeq 10 \text{ rad/s}$$

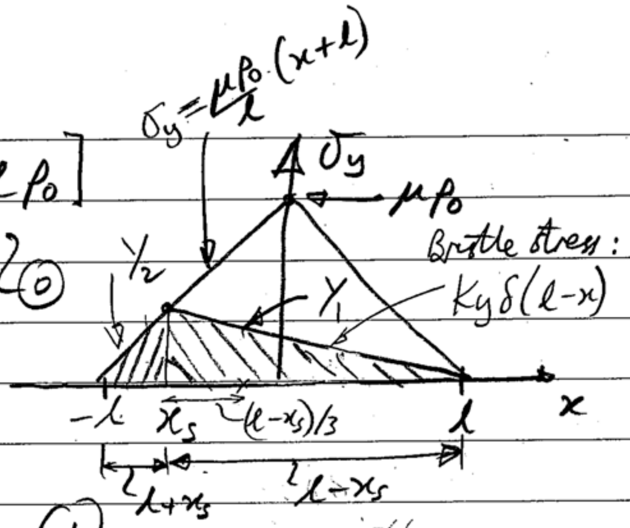


d)
- Functions demonstrate wheelbase filtering
- Bounce excitation at 10 rad/s (natural freq of vehicle) is zero, hence no bounce response at this freq.
- In contrast, pitch response will be large.
- Automotive engineers try to minimise pitch response
- Front & rear stiffnesses could be tuned to achieve this over a range of vehicle speeds.

3 (a) Vertical load: $Z = 2h\left[2 \times \frac{1}{2} l p_0\right]$

$\therefore p_0 = Z/2lh$ — (0)

(b) Microslip first occurs at the rear of the contact area at x_s when:



$$K_y\delta(l-x_s) = \mu p \quad \text{— (1)}$$

for left side of contact $\sigma_y = \frac{\mu p_0}{l}(x+l)$ — (2)

$$(1)\ \&\ (2) \Rightarrow K_y\delta(l-x_s) = \frac{\mu p_0}{l}(x_s+l)$$

$$x_s\left(\frac{\mu p_0}{l} + K_y\delta\right) = K_y\delta l - \mu p_0$$

$$\therefore x_s = \frac{l(K_y\delta l - \mu p_0)}{K_y\delta l + \mu p_0} \quad \text{— (3)}$$

Lateral force $Y = \iint \sigma_y\, dA = (\text{shaded area}) \times 2h.$

$$\therefore Y = 2h \times \frac{2l \times \sigma_y(x_s)}{2}$$

$$(2) \rightarrow \quad = 2\cancel{l}h\frac{\mu p_0}{\cancel{l}}(x_s+l)$$

$$(3) \rightarrow \quad = 2h\mu p_0 l\left[\frac{K_y\delta l - \mu p_0}{K_y\delta l + \mu p_0} + 1\right]$$

$$= 2hl\mu p_0\left[\frac{2K_y\delta l}{K_y\delta l + \mu p_0}\right] = \frac{4hl\mu p_0}{1 + \frac{\mu p_0}{K_y l\delta}} = \frac{A}{1+\frac{B}{\delta}} \quad \text{(4)}$$

$$C_\alpha = \left.\frac{\partial Y}{\partial \delta}\right|_{\delta=0} = A(\delta+B)^{-1} - A\delta(\delta+B)^{-2}\Big/_{\delta=0} = A/B$$

$$= \frac{4hl\cancel{\mu p_0}}{\cancel{\mu p_0}/K_y l} = \underline{\underline{4l^2hK_y}}$$

As per uniform contact pressure (5)

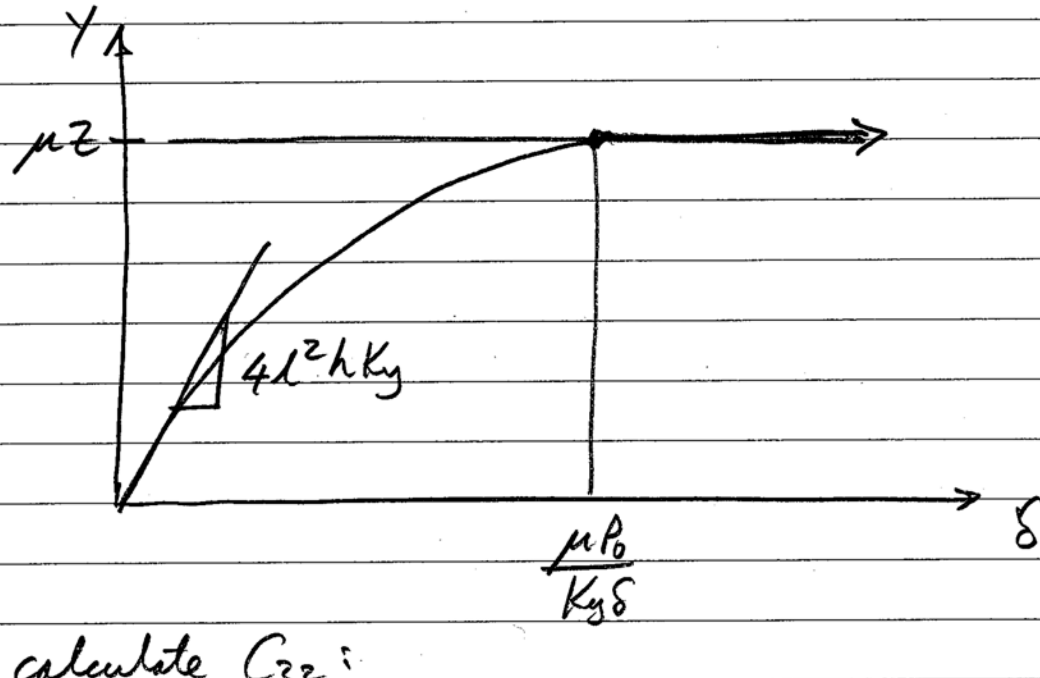
3(cont) Limiting value of δ occurs when $x_s = 0$.

$$③ \Rightarrow K_y \delta l = \mu P_0 \Rightarrow \delta = \frac{\mu P_0}{K_y l} \qquad ⑥$$

from this value of δ onwards there is full sliding with

$$④ \rightarrow \quad Y = \frac{4 l h \mu P_0}{1 + \frac{\mu P_0}{K_y \delta \left(\frac{\mu P_0}{K_y \delta}\right)}} = \underline{\underline{2 l h \mu P_0}}$$

from ⓪ this is $\underline{\underline{Y = \mu Z}}$ as expected ✓



To calculate C_{32}:

Calculate the 2 separate areas of the lateral stress graph

Y_1 = RH area × 2h. (no slip)

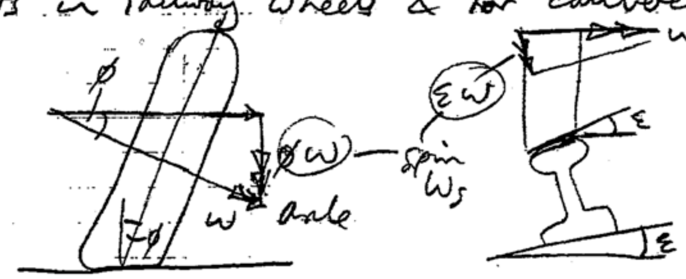
Y_2 = LH area × 2h (area of microslip)

Calculate Moment of these forces about centre of contact

$$N = Y_1 \left(\frac{l}{2} - \frac{(l - x_s)}{3} \right) + Y_2 \left[|x_s| + \frac{(l + x_s)}{3} \right]$$

$$C_{32} = \left. \frac{\partial N}{\partial \delta} \right|_{\delta = 0}$$

(b) When contact plane is not parallel to axis of rotation, the wheel has a "spin" compt of angular velocity perpendicular to contact patch — occurs in railway wheels & for cambered car tyres



(c) Cars (i) Neglect spin creep — small

(ii) Neglect realigning moment due to α since pneumatic trail is small relative to wheelbase

(iii) Neglect X — wheels free wheel or have differentials

$$\Rightarrow \underline{\underline{Y = C\alpha}}$$

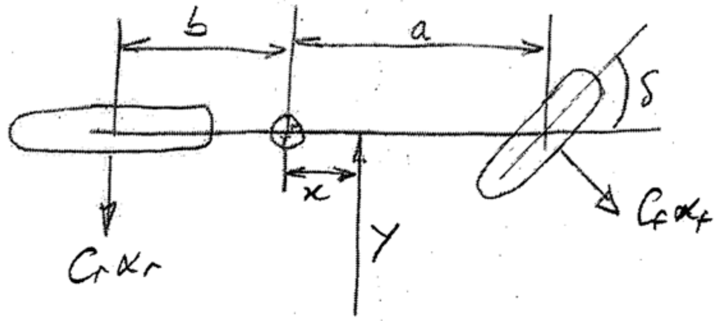
Railways (i) & (ii) as per car

(iii) Can't ignore X because two wheels on an axle don't free wheel

(iv) Reasonable to assume $K_x = K_y$

So $Y = C\alpha$ & $X = C\xi$

4. (a)



Defns

$C = C_f + C_r$

$S = \dfrac{aC_f - bC_r}{C_f + C_r}$

$q^2 = \dfrac{a^2C_f + b^2C_r}{C_f + C_r}$

$I = mk^2, \quad l = a+b$

Equations of motion: Book work - see lecture notes

$$m\begin{bmatrix}1 & 0\\ 0 & k^2\end{bmatrix}\begin{Bmatrix}\dot{v}\\ \dot{\Omega}\end{Bmatrix} + \begin{bmatrix}C/u & CS/u + mu\\ CS/u & Cq^2/u\end{bmatrix}\begin{Bmatrix}v\\ \Omega\end{Bmatrix} = \begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix} \quad \text{— (1)}$$

(b) steady state response to side wind

$\dot{v} = \dot{\Omega} = 0$ So
$$\begin{bmatrix}C/u & CS/u + mu\\ CS/u & Cq^2/u\end{bmatrix}\begin{Bmatrix}v_{ss}\\ \Omega_{ss}\end{Bmatrix} = \begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix} \quad (1a)$$

Solving for unknown steady state velocities:

$$\begin{Bmatrix}v_{ss}\\ \Omega_{ss}\end{Bmatrix} = \frac{\begin{bmatrix}Cq^2/u & -(CS/u + mu)\\ -CS/u & C/u\end{bmatrix}\begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix}}{\left(\frac{C}{u}\right)\left(\frac{Cq^2}{u}\right) - \left(\frac{CS}{u}\right)\left(\frac{CS}{u} + mu\right)} \quad \text{— (2)}$$

Multiplying out the denominator gives

$$\frac{1}{u^2}\left[C_fC_rl^2 - CSmu^2\right]$$

So
$$\frac{\Omega_{ss}}{u} = \frac{-CS(Y + C_f\delta) + C(xY + aC_f\delta)}{C_fC_rl^2 - CSmu^2} \quad \text{— (3)}$$

For $\Omega_{ss} = 0$ & $u < u_c$: $\quad \cancel{C}Y(x - S) + \cancel{C}C_f\delta(a - S) = 0$

$$\Rightarrow \delta = -\frac{(x-S)Y}{C_f(a-S)} \quad \text{— (4)}$$

[If $a = S$: $\dfrac{aC_f - bC_r}{C_f + C_r} = a \Rightarrow a\cancel{C_f} - bC_r = a\cancel{C_f} + aC_r$

$\Rightarrow (a+b)C_r = 0 \Rightarrow \underline{\underline{C_r = 0}}$

ie vehicle has only one wheel and therefore spins under the action of the yaw moment]

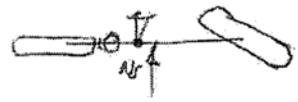
4 Cont

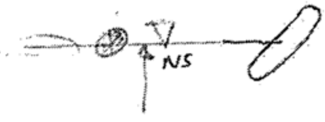
$$\delta = \frac{-(x-s)Y}{C_f(a-s)} = \frac{(C_f+C_r)(s-x)}{C_f C_r \cdot L} \quad \text{for } \Omega_{ss}=0$$

for $s<0$ (understeer) $a-s>0$

for $s>0$ (oversteer) $|s|<a$ ie NS point is always within the frame $\Rightarrow$ so denominator is always +ve

If $x=s$, $\delta=0$ ie the force is applied at the neutral steer point

If $x>s$ $\delta<0$ ie turn into wind 

If $x<s$ $\delta>0$ ie turn away from wind 

(c) To find sideslip, substitute (4) into (1a), with $\Omega_{ss}=0$:

$$C/u \, v_{ss} + (Cs/u + mu)\underbrace{\cancel{\Omega_{ss}}}_{=0} = Y + C_f\delta_{ss}$$

$$\frac{v_{ss}}{u} = \beta_{ss} \Rightarrow C\beta_{ss} = Y + \frac{\cancel{C_f}(s-x)Y}{\cancel{C_f}(a-s)} = \frac{Y(a-s+s-x)}{a-s}$$

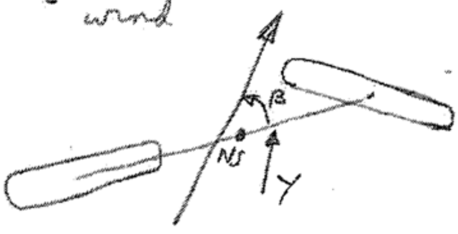
$$\text{So} \quad \frac{C\beta_{ss}}{Y} = \frac{a-x}{a-s}$$

x can't be $>a$ because the force would be applied forward of the front wheel. So numerator is always >0

Denominator is always >0 because $|s|<a$

Conclusion: $\frac{C\beta_{ss}}{Y}>0$ for all conditions ie the vehicle always has a steady sideslip velocity away from the wind

$x>s$



$x<s$