Part IIB Module 4C8 Solutions, 2024

1 (a)

The key assumptions are:

- Vibration only occurs in vertical direction.
- Displacement z_s and z_u are deviations from static equilibrium positions.
- Contact with road is at a point with no loss of contact.
- Suspension is modelled as a parallel linear spring-damper with no friction.
- Tyre is modelled as a linear spring with no vertical damping.

1 (b)

The equations of motion of the quarter-car model can be expressed as

$$m_s \ddot{z}_s = -k(z_s - z_u) - c(\dot{z}_s - \dot{z}_u) \tag{1.1}$$

$$m_u \ddot{z}_u = k(z_s - z_u) + c(\dot{z}_s - \dot{z}_u) - k_t(z_u - z_r) \tag{1.2}$$

Applying Laplace transforms to Eqs. (1.1) and (1.2) in zero initial conditions gives

$$s^2 m_s z_s(s) = -k\left[z_s(s) - z_u(s)\right] - sc\left[z_s(s) - z_u(s)\right] \tag{2.1}$$

$$s^2 m_u z_u(s) = k\left[z_s(s) - z_u(s)\right] + sc\left[z_s(s) - z_u(s)\right] - k_t\left[z_u(s) - z_r(s)\right] \tag{2.2}$$

Transfer function $z_s(s) / z_u(s)$ can be obtained by rearranging Eq. (2.1):

$$\frac{z_s(s)}{z_u(s)} = \frac{sc + k}{s^2 m_s + sc + k} \tag{3}$$

Note that Eqs. (1.2) and (2.2) are not needed to derive the transfer function requested for this part. But they will be used to solve part (e) later.

1 (c)

- Body acceleration (BA) is quantified using the acceleration magnitude of the sprung mass in response to road input. It is a measure of ride comfort/discomfort.
- Suspension working space (WS) is quantified using the relative displacement of the sprung and unsprung masses. It is a measure of the vertical space required in the suspension system. There is usually limited space available for relative displacements between sprung and unsprung masses in road vehicles.
- Tyre dynamic force (TF) is quantified using the vertical force applied to the tyre by road. It is a measure of the ability of the tyres to generate accelerating, braking or cornering force. This ability is reduced if the vertical tyre force oscillates.

1 (d) (i)

- κ indicates the roughness of road. Larger κ means higher roughness.
- w indicates the downwards 'slope' of the MSSD. It captures how fast $S_{z_r}(n)$ decreases as wave number n increases. A typical value of w is 2.5.

1 (d) (ii)

Following lecture notes, the following expression holds

$$E\left[z_r^{\,2}\right] = \int_{n=0}^{n=\infty} S_{z_r}(n)\mathrm{d}n = \int_{\omega=0}^{\omega=\infty} S_{z_r}(\omega)\mathrm{d}\omega \tag{4}$$

Eq. (4) is derived from the fact that changing from wavenumber domain to frequency domain does not alter the mean square value of road displacement input z_r.

Substituting $S_{z_r}(n) = \kappa n^{-w}$ into (4) gives

$$E\left[z_r^{\ 2}\right] = \int_{n=0}^{n=\infty} \kappa n^{-w} \mathrm{d}n = \int_{\omega=0}^{\omega=\infty} S_{z_r}(\omega)\mathrm{d}\omega \tag{5}$$

Recall that the relationship between wavenumber n and angular frequency ω is $\omega = 2\pi nV$, where V is vehicle speed. Substituting this into (5) to eliminate n yields

$$E\left[z_r^{\ 2}\right] = \int_{\omega=0}^{\omega=\infty} \kappa(\frac{\omega}{2\pi V})^{-w} \mathrm{d}(\frac{\omega}{2\pi V}) = \int_{\omega=0}^{\omega=\infty} S_{z_r}(\omega)\mathrm{d}\omega \tag{6}$$

Eq. (6) suggests that

$$S_{z_r}(\omega) = \kappa(2\pi V)^{w-1}\omega^{-w} \tag{7}$$

1 (d) (iii)

For $w = 2$, Eq. (7) becomes

$$S_{z_r}(\omega) = 2\pi\kappa V\omega^{-2} \tag{8}$$

Following lecture notes, $S_{\dot{z}_r}(\omega)$ can be related to $S_{z_r}(\omega)$ using:

$$S_{\dot{z}_r}(\omega) = \omega^2 S_{z_r}(\omega) \tag{9}$$

Substituting (8) into (9) gives

$$S_{\dot{z}_r}(\omega) = 2\pi\kappa V \tag{10}$$

Using Eqs. (9) and (10), $S_{z_r}(\omega)$ and $S_{\dot{z}_r}(\omega)$ can be then sketched in a log-log graph, as shown in Fig. C0.

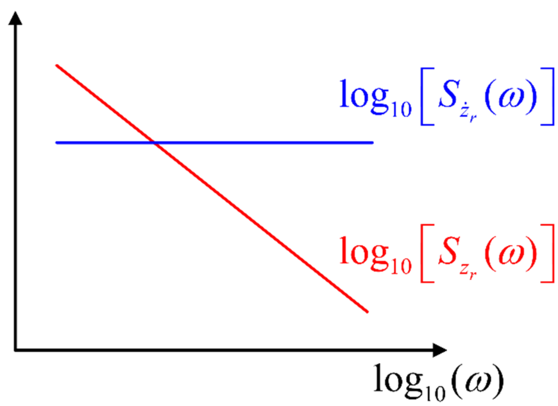


Fig. C0

It can be seen from Eq. (10) or the sketch of $S_{\dot{z}_r}(\omega)$ that for $w = 2$, the MSSD of road vertical velocity profile is a white noise. Using $S_{\dot{z}_r}(\omega)$ allows convenient and efficient calculation of vehicle response and evaluation of performance, in comparison to using the $S_{z_r}(\omega)$.

1 (e)

Let's start with the expression requested, which is rewritten as below:

$$m_s H_{BA}(s) - \left(1 + \frac{m_u}{k_t}s^2\right)H_{TF}(s) + m_u s = 0 \tag{11}$$

Substituting the expressions for $H_{BA}(s)$ and $H_{TF}(s)$ given in part (c) into Eq. (11) yields

$$m_s \frac{\ddot{z}_s(s)}{\dot{z}_r(s)} - \left(1 + \frac{m_u}{k_t} s^2\right) \frac{k_t[z_r(s) - z_u(s)]}{\dot{z}_r(s)} + m_u s = 0 \tag{12}$$

Multiplying Eq. (12) by $\dot{z}_r(s)$ and taking $\ddot{z}_s(s) = s^2 z_s(s)$ and $\dot{z}_r(s) = s z_r(s)$ then gives

$$m_s s^2 z_s(s) - k_t[z_r(s) - z_u(s)] + s^2 m_u z_u(s) = 0 \tag{13}$$

It can be seen that Eq. (13) is the sum of equations of motions (2.1) and (2.2) derived for part (b). Hence, the derivation of Eq. (11) can be formally formulated by combining Eqs. (2.1) and (2.2) to give Eq. (13) in the first place and then diving Eq. (13) by $\dot{z}_r(s)$ and substituting $H_{BA}(s)$ and $H_{TF}(s)$ to finally give Eq. (11).

To find the angular frequency at which $H_{BA}(s)$ is independent of $H_{TF}(s)$, we first need to replace the Laplace transform variable *s* in Eq. (11) with $j\omega$. This gives

$$m_s H_{BA}(j\omega) - \left(1 - \frac{m_u}{k_t} \omega^2\right) H_{TF}(j\omega) + j\omega m_u = 0 \tag{14}$$

It can be seen from Eq. (14) that $H_{BA}(s)$ is independent of $H_{TF}(s)$ when

$$1 - \frac{m_u}{k_t}\omega^2 = 0\text{, or equivalently } \omega = \sqrt{\frac{k_t}{m_u}}.$$

At this frequency, $H_{BA}(s)$, or $H_{BA}(j\omega)$ is only dependent of sprung mass m_s and unsprung mass m_u but cannot be influenced by suspension stiffness *k* or damping *c*.

Regarding the trade-off between root-mean-square values of BA and TF, it can be seen from Eq. (11) that once $H_{BA}(s)$ is specified to achieve optimal driving comfort, $H_{TF}(s)$ is set down by Eq. (11). This resultant $H_{TF}(s)$ is usually not the optimum from the perspective of maximising tyre road contact. This trade-off is quantitatively represented in the 'conflict diagram' of RMS responses discussed in the lecture notes.

2 (a)

The force-velocity characteristics of the damper is sketched in Fig. C1. Note that the sketch should include the following three key features:

- The curve dominated by the fixed orifice,
- The curve controlled by the spring inside the damper, and
- Damping in the compression regime is less than in the extension – to avoid large damping force when the wheel his bumps on the road.

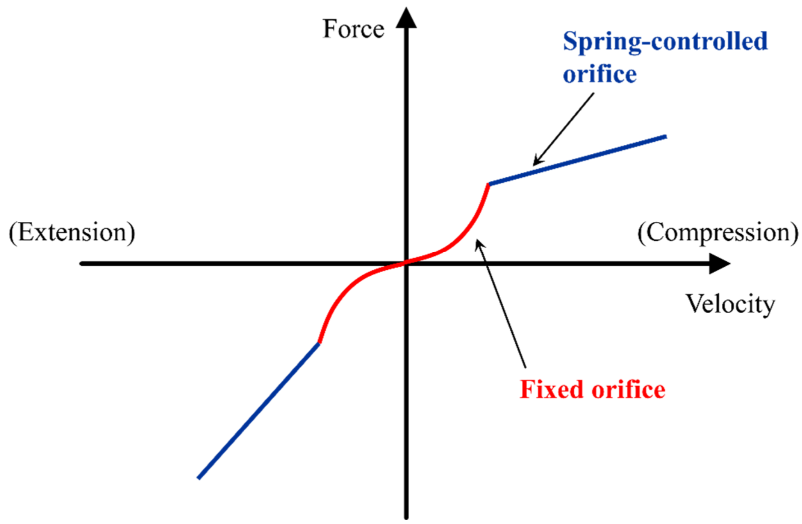


Fig. C1

2 (b)

Following lecture notes, tyre lateral force F_{lat} in the series spring-damper model can be expressed as

$$F_{lat} + \frac{c_{lat}}{k_{lat}} \dot{F}_{lat} = c_{lat} \dot{y} \tag{1}$$

Eq. (1) is a first-order differential equation. Hence, the time constant, by definition, is $\tau_{lat} = c_{lat} / k_{lat}$.

2 (c)

The 4-DOF lateral-roll model is sketched in Fig. C2, where the four degrees of freedom are: sprung mass rotation θ_s, unsprung mass rotation θ_u, and tyre model lateral motion y and y_t. suspension roll stiffness τ_s, suspension roll damping η_s and tyre to ground roll stiffness τ_t are annotated in Fig. C2. It is important to annotate that ϕ denotes the roll input from road surface.

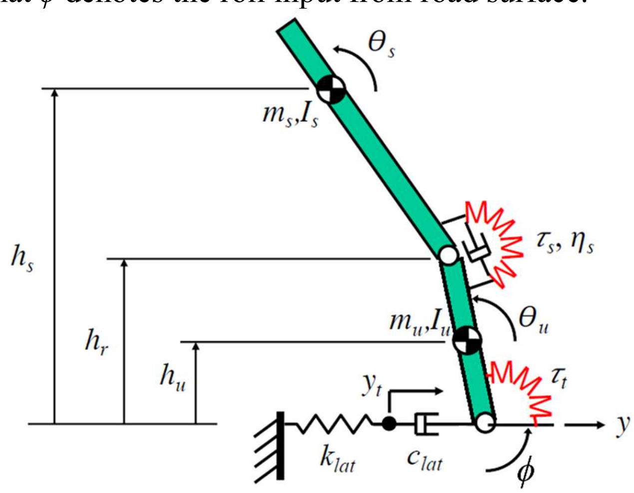


Fig. C2

2 (d) (i)
For vehicles with very large roll stiffness τ_s, sprung mass m_s and unsprung mass m_u will experience the same roll angle θ during roll motion.

2 (d) (ii)
Note that tyre lateral stiffness c_{lat} is defined as $c_{lat} = C / U$, where C is tyre cornering stiffness. C is also known as tyre lateral creep coefficient or side-force coefficient. Therefore, for high vehicle speed $c_{lat} \rightarrow 0$. This is equivalent to removing the damper in the series spring-damper tyre model. As a result, spring k_{lat} is no longer connected to the unsprung mass m_u. A sketch of the tyre model in this situation is given in Fig. C3 (a). Because spring k_{lat} is no longer connected to the unspring mass m_u, y_t no longer affects y.

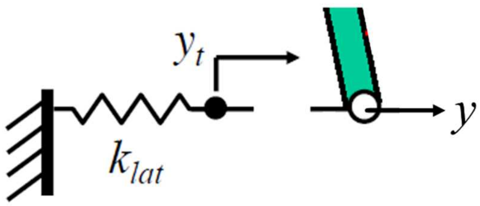


Fig. C3 (a)

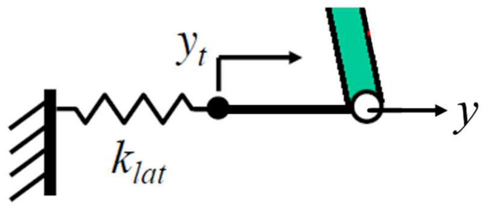


Fig. C3 (b)

By the same token, for nearly zero vehicle speed $c_{lat} \rightarrow \infty$. This is equivalent to locking the damper in the series spring-damper tyre model. A sketch of the resultant tyre model is given in Fig. C3 (b). In this case, y_t and y represents the same motion. The force applied to unsprung mass m_u is $-k_{lat}\, y$.

2 (e) (i)
The 2-DOF model in the condition of nearly zero vehicle speed is sketched in Fig. C4. Note that in this model the sprung and unsprung masses have been lumped into a single vehicle body mass m. The tyre model now has one degree of freedom y due to $c_{lat} \rightarrow \infty$.

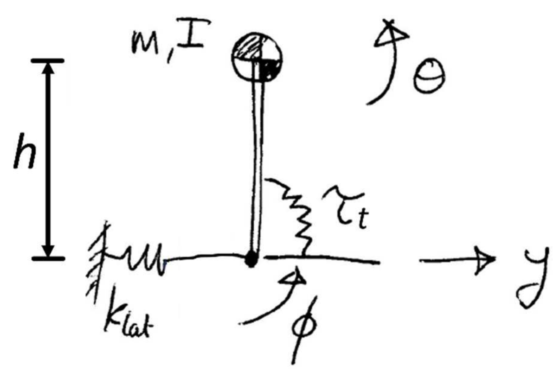


Fig. C4

The equations of motion can be expressed as

$$I\ddot{\theta} = -\tau_t\theta - hk_{lat}y + \tau_t\phi \tag{2.1}$$
$$m(\ddot{y} - h\ddot{\theta}) = -k_{lat}y \tag{2.2}$$

In Eq. (2.1), $-\tau_t\theta$ is the torque applied by the torsion spring τ_t (representing tyre to ground roll stiffness) to vehicle body due to body roll θ. $-hk_{lat}y$ is the moment of tyre lateral force $-k_{lat}\, y$ about vehicle centre of mass. $+\tau_t\phi$ is the torque generated due to the road surface roll input ϕ. In Eq. (2.2), $\ddot{y} - h\ddot{\theta}$ is the acceleration at vehicle centre of mass.

Expressing Eqs. (2.1) and (2.2) in matrix form gives

$$\begin{bmatrix} I & 0 \\ -mh & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} \tau_t & k_{lat}h \\ 0 & k_{lat} \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{bmatrix} \tau_t \\ 0 \end{bmatrix} \phi \tag{3}$$

2 (e) (ii)

To calculate undamped natural frequencies, zero input is assumed. As a result, Eq. (3) becomes

$$\begin{bmatrix} I & 0 \\ -mh & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} \tau_t & k_{lat}h \\ 0 & k_{lat} \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4}$$

Applying Laplace transform to (4) in zero initial conditions, and then replacing Laplace transform variable s with $j\omega$ gives

$$-\omega^2 \begin{bmatrix} I & 0 \\ -mh & m \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} + \begin{bmatrix} \tau_t & k_{lat}h \\ 0 & k_{lat} \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

Undamped natural frequency can be determined by setting

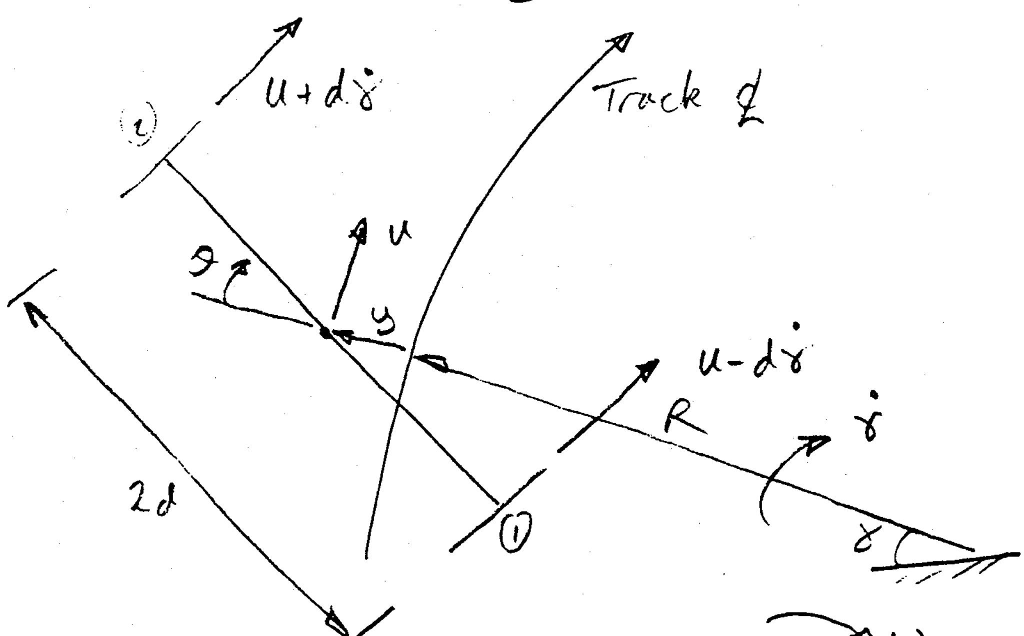
$$\begin{vmatrix} \tau_t - I\omega^2 & k_{lat}h \\ mh\omega^2 & k_{lat} - m\omega^2 \end{vmatrix} = 0$$

which gives

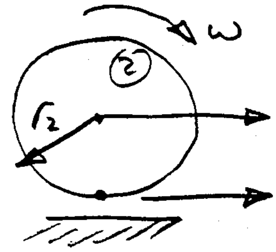
$$mI\omega^4 - \omega^2 (k_{lat} I + m\tau_t + mh^2) + k_{lat}\tau_t = 0 \tag{6}$$

Undamped natural frequencies can be then calculated using quadratic formula.

Q3 (a) In a steady turn, $\dot{y} = \dot{\theta} = 0$. The velocity components on a single wheelset are:



Longitudinal creep



$\rightarrow u + d\dot{\gamma}$

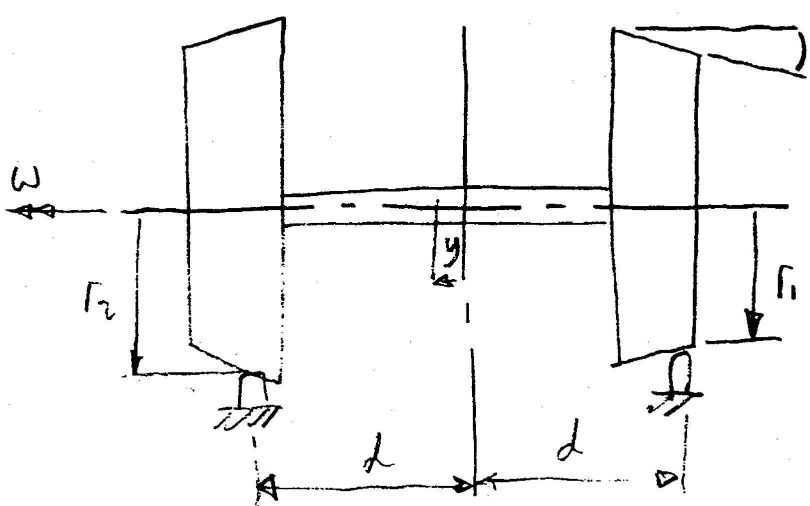
$V_x = u + d\dot{\gamma} - r_2\omega$ (= 0 if no slip)

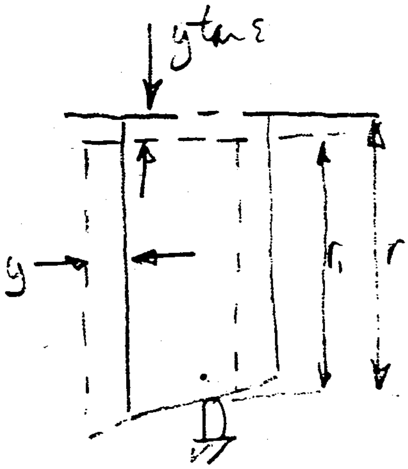
Angular vel:

$$\omega = \frac{\text{average vel}}{\text{average rolling rad}} = \frac{u}{r} \qquad \text{—— (1)}$$

$$\dot{\gamma} = \frac{u}{R+y} \simeq \frac{u}{R} \qquad \text{—— (2)}$$

Rolling radii





$$\left.\begin{aligned} r_1 &= r - y\tan\varepsilon \simeq r - y\varepsilon \\ r_2 &= r + y\tan\varepsilon \simeq r + y\varepsilon \end{aligned}\right\} \;($$

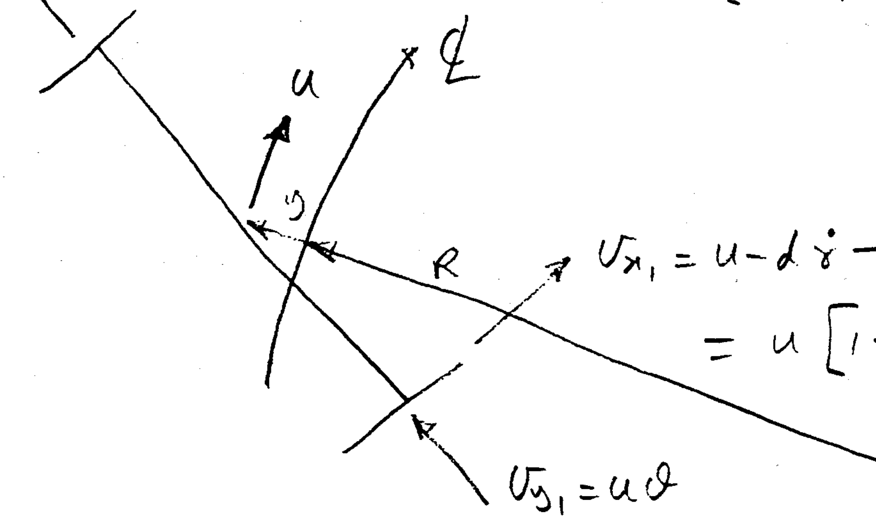
3 cont

Creep vels:

$v_{y_2} = u\theta$

$v_{x_2} = u + d\dot{\gamma} - r_2\omega = u\left[1 + d/R - \left(1 + \frac{\varepsilon y}{r}\right)\right]$ (

(from (2)) (from (1) & (3))

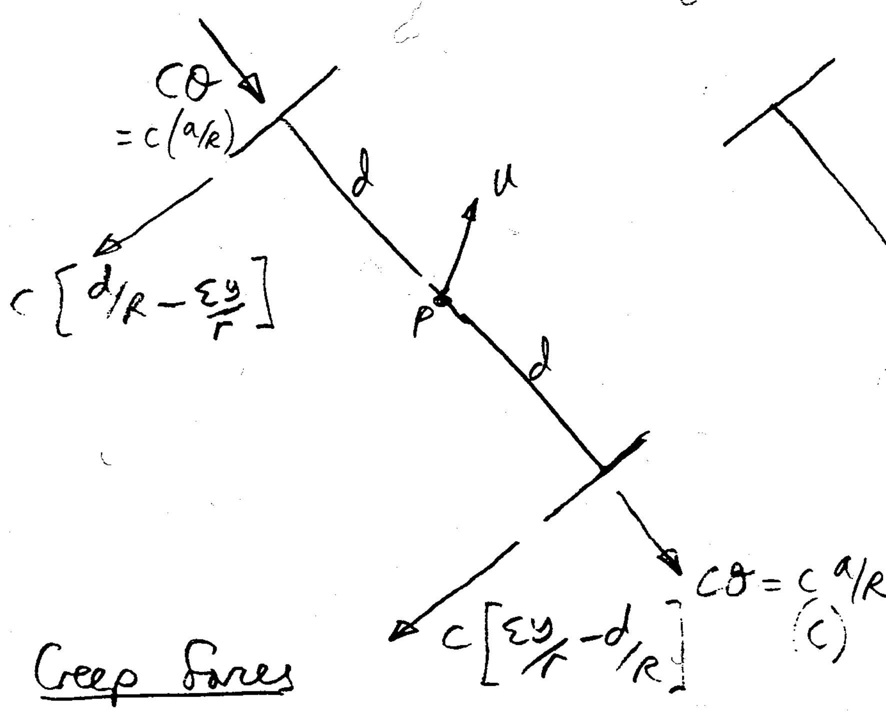


$v_{x_1} = u - d\dot{\gamma} - r_1\omega$

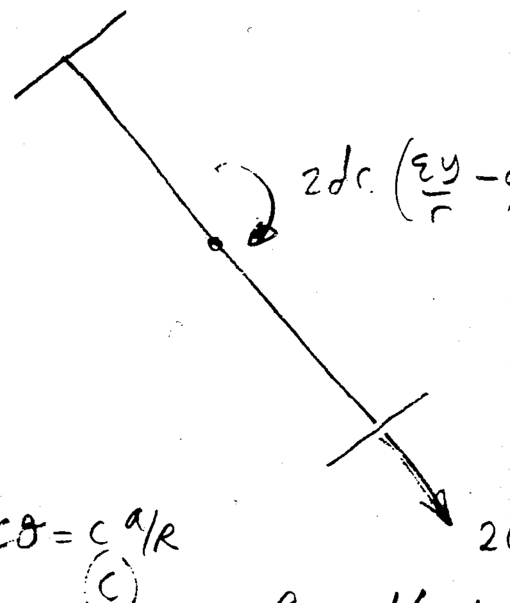
$= u\left[1 - d/R - \left(1 - \frac{\varepsilon y}{r}\right)\right]$

$v_{y_1} = u\theta$

Creep forces are $F_x = -C\frac{v_x}{u}$ & $F_y = -C\frac{v_y}{u}$ (



Creep forces



Resultants

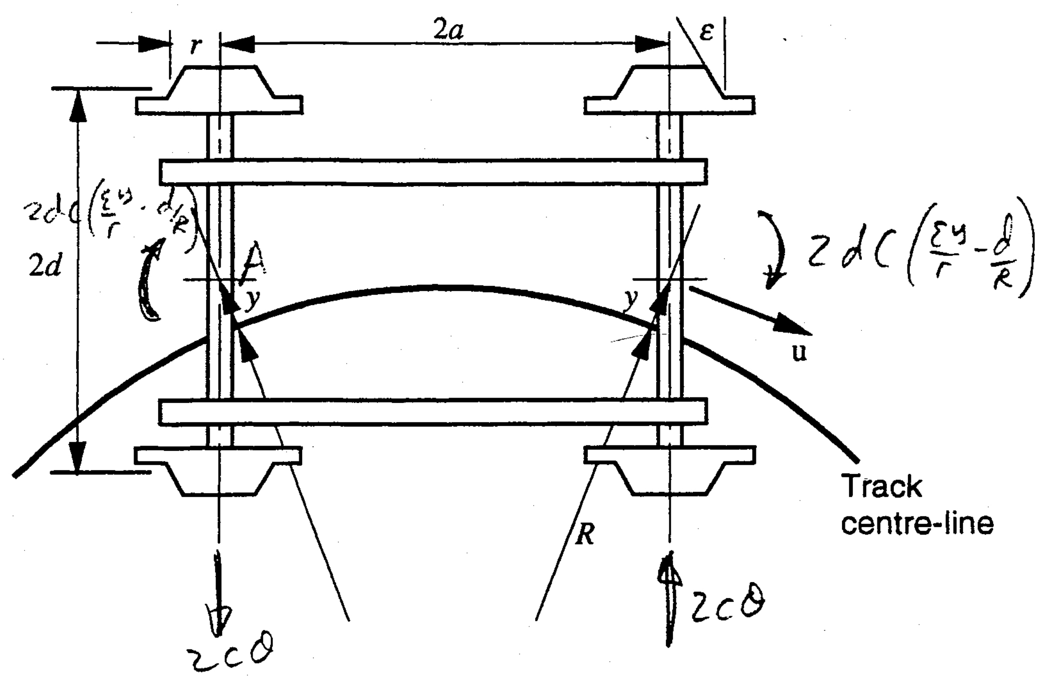
Resultant moment at P is $2dc\left(\frac{\varepsilon y}{r} - \frac{d}{R}\right)$ (7

[4

(b) Assumptions

(i) Motion is well damped
(ii) Equal lateral & long'l creep coeffs
(iii) Neglect spin creep & realigning moments
(iv) Centrifugal forces balanced by normal reactions due to cant angle of track

3 cont



$$\circlearrowright + \quad \Sigma M_A \quad 4dc\left(\frac{\varepsilon y}{r} - \frac{d}{R}\right) - 2a\left(2c\theta\right) = 0 \qquad (8)$$

Also $\theta = a/R$ (9)

eliminate θ from 8 & solve for y:

$$\boxed{y = \frac{dr}{\varepsilon R}\left(1 + a^2/d^2\right)} \qquad (10) \quad [30]$$

(c) Consider the outer wheel:

from (10)

$$F_{x_2} = c\left(d/R - \varepsilon y/r\right) = c\left[\frac{d}{R} - \frac{rd}{\varepsilon R}\left(1 + \frac{a^2}{d^2}\right)\left(\frac{\varepsilon}{r}\right)\right]$$

$$= -\frac{ca^2}{dR} \qquad \left(\text{NB inner wheelset gives } + ca^2/dR\right)$$

Resultant creep force is:

$$F_{tot} = \sqrt{(c\theta)^2 + \left(\frac{ca^2}{dR}\right)^2} = \frac{ca}{R}\sqrt{1 + \left(\frac{a}{d}\right)^2}$$

Max total creep force is $\mu Z/2$ ∴ vehicle slides when

$$\frac{\mu Z}{2} = \frac{ca}{R_{lim}}\sqrt{1 + a^2/d^2} \Rightarrow R_{lim} = \frac{2ca}{\mu Z}\sqrt{1 + \frac{a^2}{d^2}}$$

Flange contact thereafter [30]

4. (a) See Lecture Notes for derivation of equations of motion (40%)

(b) (i) $\delta = -K\theta \qquad \Omega = \dot{\theta}$ ie $\bar{\Omega} = s\bar{\theta}$ (s = Laplace T/F)

So the eq's of motion are

$$m(\dot{v} + u\Omega) + (C_f + C_r)\frac{v}{u} + (aC_f - bC_r)\frac{\Omega}{u} + C_f\frac{K}{s}\Omega = 0$$

and

$$I\dot{\Omega} + (aC_f - bC_r)\frac{v}{u} + (a^2C_f + b^2C_r)\frac{\Omega}{u} + aC_f\frac{K}{s}\Omega = 0$$

putting $\bar{\dot{v}} = s\bar{v}$ & $\bar{\dot{\Omega}} = s\bar{\Omega}$ gives the characteristic eqn:

$$\begin{vmatrix} ms + \frac{C_f + C_r}{u} & mu + \left(\frac{aC_f - bC_r}{u}\right) + \frac{C_f K}{s} \\ \frac{aC_f - bC_r}{u} & Is + \left(\frac{a^2C_f + b^2C_r}{u}\right) + \frac{aC_f K}{s} \end{vmatrix} \begin{matrix} \bar{v} \\ \bar{\Omega} \end{matrix} = 0$$

Hence

$$\left[ms + \frac{C_f + C_r}{u}\right]\left[Is + \left(\frac{a^2C_f + b^2C_r}{u}\right) + \frac{aC_f K}{s}\right] - \left[\frac{aC_f - bC_r}{u}\right]\left[mu + \left(\frac{aC_f - bC_r}{u}\right) + \frac{C_f K}{s}\right] = 0$$

ie

$$\left[ms^2 + \left(\frac{C_f + C_r}{u}\right)s\right]\left[Is^2 + \left(\frac{a^2C_f + b^2C_r}{u}\right)s + aC_f K\right] - \left[\left(\frac{aC_f - bC_r}{u}\right)s\right]\left[mus + \left(\frac{aC_f - bC_r}{u}\right)s + C_f K\right] = 0$$

$$mIs^3 + \left[\frac{(a^2C_f + b^2C_r)m + (C_f + C_r)I}{u}\right]s^2 + \left[\frac{(C_f + C_r)(a^2C_f + b^2C_r)}{u^2} + maC_f K - \left(\frac{aC_f - bC_r}{u^2}\right)^2 - (aC_f - bC_r)m\right]s^1 + \left[\left(\frac{C_f + C_r}{u}\right)aC_f K - \left(\frac{aC_f - bC_r}{u}\right)C_f K\right]s = 0$$

(30%)

Neglect $s = 0$ soln $\Rightarrow a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

4. Use the Routh - Hurwitz Criteria in the data book

(i) All a's > 0

(ii) $a_1 a_2 > a_0 a_3$

(10%)

Substitute for a's from Characteristic eqn.
→ obtain inequalities for K.

(iii) This ~~strategy~~ ensures that the vehicle remains parallel with the direction of motion ($\theta = 0$) but does not control the sideslip - so the lateral position can increase uncontrolled.
∴ Need another control loop with $-K' \frac{U}{s}$ (20%)