EGT3
ENGINEERING TRIPOS PART IIB

Thursday 25 April 2024 9.30 to 11.10

Module 4C8

VEHICLE DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Single-sided script paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed.

Attachment: 4C8 data sheet, 2023 (3 pages).

Engineering Data Book.

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A quarter-car model is shown in Fig. 1. The sprung mass is m_s and the unsprung mass is m_u. Their corresponding displacements (from vehicle stationary position) are z_s and z_u. The suspension stiffness and damping are *k* and *c*, respectively. The tyre vertical stiffness is k_t. The road input displacement is z_r.

(a) State the key assumptions implicit in the model. [10%]

(b) Show that transfer function $z_s(s)/z_u(s)$ takes the following form, assuming zero initial conditions:

$$\frac{z_s(s)}{z_u(s)} = \frac{sc + k}{s^2 m_s + sc + k}$$

where *s* is the Laplace transform variable. [15%]

(c) Transfer functions relating road input velocity to three output quantities: body acceleration (BA), suspension working space (WS), and dynamic tyre force (TF) are defined as:

$$H_{BA}(s) = \frac{\ddot{z}_s(s)}{\dot{z}_r(s)}, \quad H_{WS}(s) = \frac{z_s(s) - z_u(s)}{\dot{z}_r(s)} \quad \text{and} \quad H_{TF}(s) = \frac{k_t[z_r(s) - z_u(s)]}{\dot{z}_r(s)}.$$

Briefly state the relevance of the three output quantities to the performance of a vehicle. [15%]

(d) The single-sided mean square spectral density (MSSD) of the vertical displacement along one wheel-track of a randomly rough surface can be modelled as

$$S_{z_r}(n) = \kappa n^{-w}$$

where *n* is the road profile wavenumber.

(i) Explain the significance of the parameters κ and *w*. [10%]

(ii) Explain how $S_{z_r}(\omega)$ can be derived from $S_{z_r}(n)$, where ω is the angular frequency. [15%]

(iii) For a special case of $w = 2$, sketch $S_{z_r}(\omega)$ as a function of ω on a log-log plot. Sketch the single-sided MMSD of the corresponding road input velocity $S_{\dot{z}_r}(\omega)$ on the same graph. Use the sketches to explain why road input velocity $\dot{z}_r$, as opposed to displacement z_r is a more convenient input quantity for evaluating vehicle performance in terms of BA, WS and TF. [20%]

(e) Use some or all of the expressions given in (c) to show that

$$m_s H_{BA}(s) - \left(1 + \frac{m_u}{k_t} s^2\right) H_{TF}(s) + m_u s = 0.$$

Find an angular frequency ω at which H_{BA} is independent of H_{TF}. Comment on how the trade-off between root-mean-square values of BA and TF can be understood using the expression given in (e). [15%]

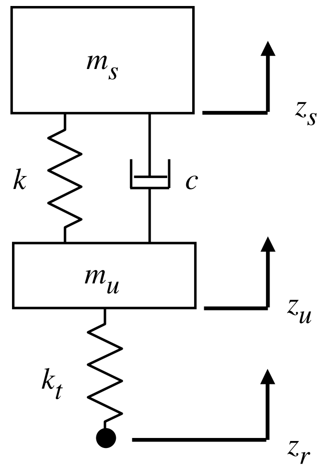


Fig. 1

2 A roll-plane model of a vehicle is shown in Fig. 2. The model has six degrees of freedom: $z_s, z_u, \theta_s, \theta_u, y$ and y_t. The sprung mass m_s has roll inertia I_s about its centre of mass. The unsprung mass m_u has roll inertia I_u about its centre of mass. The height of the centre of gravity of the sprung mass above the road surface is h_s and that of the unsprung mass is h_u. The sprung mass is assumed to rotate about a roll centre of height h_r. The suspension stiffness and damping are k_s and c_s, respectively. The tyre vertical stiffness is k_t. The lateral dynamics of the tyre is represented by a series spring of stiffness k_{lat} and damper of damping c_{lat}. The distance between the two suspensions is 2*S*, and the distance between the two tyres is 2*T*. The vehicle travels at speed *U* along a randomly rough road surface which has vertical displacements z_L and z_R on the left and right tracks.

(a) The suspension has monotube dampers, each with one fixed orifice and several spring-controlled orifices. Sketch the force-velocity characteristics of the dampers. Annotate the compression and extension regimes in your sketch. [10%]

(b) Show that the time constant of the series spring-damper lateral tyre model is:

$$\tau_{lat} = c_{lat}/k_{lat}.$$ [20%]

(c) The six-degree-of-freedom (6-DOF) model shown in Fig. 2 may be simplified to a four-degree-of-freedom (4-DOF) model that represents only the lateral-roll vibration of the vehicle. Sketch the simplified model and annotate the four degrees of freedom and the road input. You may use the following parameters in your sketch: suspension roll stiffness $\tau_s = 2k_sS^2$, suspension roll damping $\eta_s = 2c_sS^2$, and tyre to ground roll stiffness $\tau_t = 2k_tT^2$. [10%]

(d) The 4-DOF model may be further simplified to a two-degree-of-freedom (2-DOF) model, where the sprung mass m_s and unsprung mass m_u experience the same roll angle θ, and the lateral tyre dynamics can be described using one degree of freedom.

(i) For what type of vehicle can it be assumed that masses m_s and m_u will have the same roll angle θ. [10%]

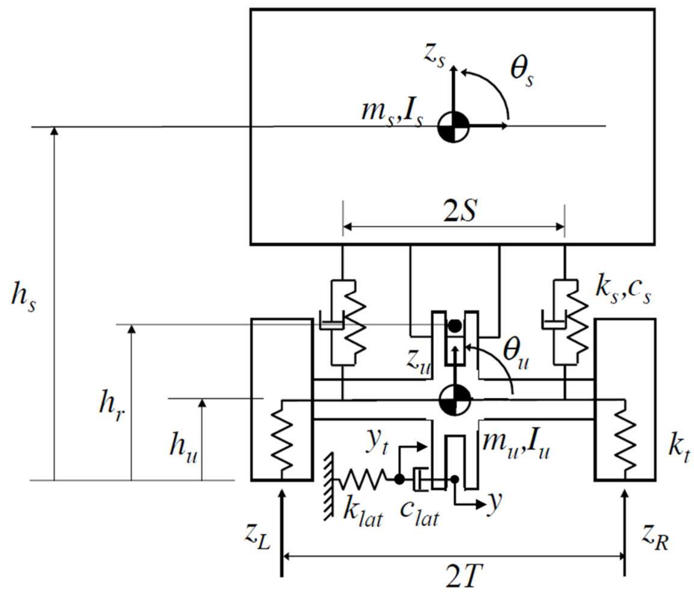
(ii) Explain with the aid of sketches, how the lateral tyre dynamics model can be simplified for the cases of very high vehicle speed and near zero vehicle speed. [10%]

(e) The 2-DOF model discussed in (d) has mass *m* and roll inertia *I* about centre of mass. The height of the centre of mass is *h*.

(i) For the case of nearly zero vehicle speed, express the equations of motion in matrix form. [20%]

(ii) Calculate the undamped natural frequencies of vibration for this case. [20%]

 (cont.



z_s
θ_s
m_s, I_s
$2S$
h_s
k_s, c_s
h_r
z_u
θ_u
h_u
y_t
m_u, I_u
k_t
k_{lat}
c_{lat}
y
z_L
z_R
$2T$

Fig. 2

3 Figure 3 shows a light, rigid, two-axle railway bogie of length 2*a* which is moving at steady speed *u* on a curved track of large constant radius *R* and gauge 2*d*. The effective conicity of the wheels is ϵ and the average rolling radius of the wheels is *r*. The coefficients of lateral and longitudinal creep of the wheels are both *C*. The lateral tracking error is *y*. The curvature of the track and the tracking error are exaggerated in the figure.

(a) Assuming that there is no oscillatory lateral or yawing motion, show that the steady state longitudinal and lateral forces on each wheel are given by: [30%]

$$F_x = \pm C\left(\frac{d}{R} - \frac{y\epsilon}{r}\right)$$
$$F_y = \pm C\frac{a}{R}$$

(b) Derive an expression for *y* in terms of the geometric variables. State your assumptions. [30%]

(c) Under these conditions of steady cornering calculate the total creep force at each wheel. [20%]

(d) When a normal load *Z* is carried by each axle, calculate the smallest curve radius that the bogie can negotiate without sliding. Take the coefficient of friction as μ and assume that the lateral and longitudinal creep forces remain linear with creep velocity right up to the onset of sliding. Comment on the validity of this assumption. [20%]

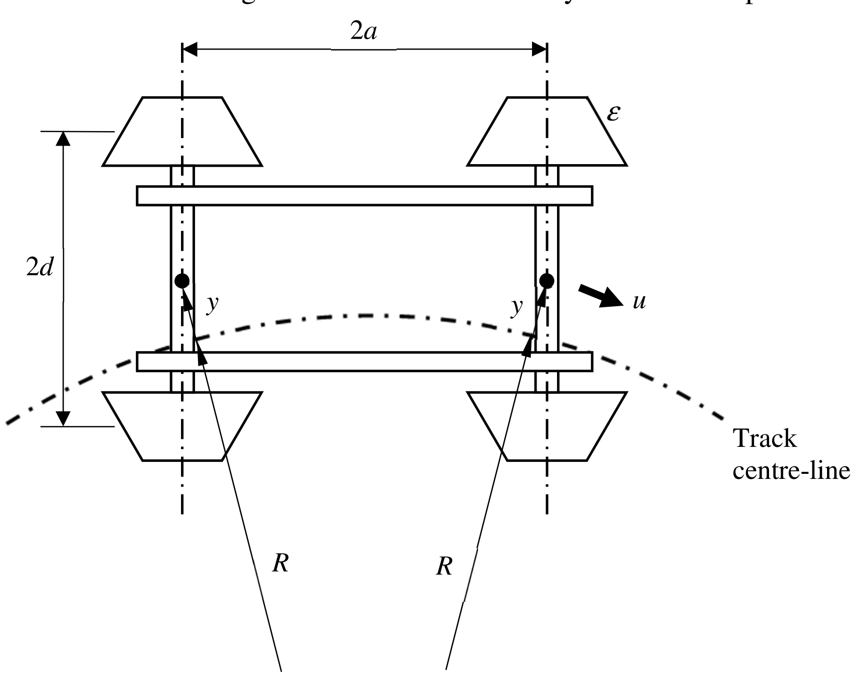


Fig. 3

4 (a) A 'bicycle' model of a car, with freedom to sideslip with velocity v and yaw at rate Ω, is shown in Fig. 4. The car moves at steady forward speed u. It has mass m, yaw moment of inertia I, and lateral creep coefficients of C_f and C_r at the front and rear axles. The lengths a and b and the steering angle δ are defined in the figure. Show that the equations of motion in a coordinate frame rotating with the vehicle are given by:

$$m(\dot{v} + u\Omega) + \left(C_f + C_r\right)\frac{v}{u} + \left(aC_f - bC_r\right)\frac{\Omega}{u} = C_f\delta$$

$$I\dot{\Omega} + \left(aC_f - bC_r\right)\frac{v}{u} + \left(a^2C_f + b^2C_r\right)\frac{\Omega}{u} = aC_f\delta$$

State your assumptions. [40%]

(b) An automatic rapid transit vehicle is designed to follow a straight path at speed u. A control system steers the front wheels in response to angular deviations of the vehicle from the direction of motion θ, so that $\delta = -K\theta$, where K is a constant.

(i) Derive a characteristic equation for small deviations in trajectory from a straight path. [30%]

(ii) Explain how you would use the Routh-Hurwitz criteria to find the range of K for which the motion of the vehicle is stable. [10%]

(iii) Comment briefly on the effectiveness of this control strategy. [20%]

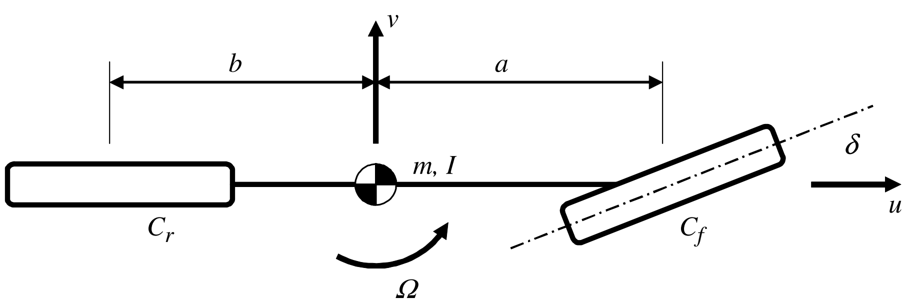


Fig. 4

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Engineering Tripos Part IIB

Data sheet for Module 4C8: Applications of Dynamics

DATA ON VEHICLE DYNAMICS

1. Creep Forces In Rolling Contact

1.1 Surface tractions

Longitudinal force $X = \iint_A \sigma_x \, dA$

Lateral force $Y = \iint_A \sigma_y \, dA$

Realigning Moment $N = \iint_A \left(x\,\sigma_y - y\,\sigma_x \right) dA$

where

σ_x, σ_y = longitudinal, lateral surface tractions

x, y = coordinates along, across contact patch

A = area of contact patch

1.2 Brush model

$\sigma_x = K_x q_x,\ \sigma_y = K_y q_y$ for $\sqrt{\sigma_x^2 + \sigma_y^2} \le \mu p$

where

q_x, q_y = longitudinal, lateral displacements of 'bristles' relative to wheel rim

K_x, K_y = longitudinal, lateral stiffness per unit area

μ = coefficient of friction

p = local contact pressure

1.3 Linear creep equations

$X = -C_{11}\xi$

$Y = -C_{22}\alpha + C_{23}\psi$

$N = C_{32}\alpha + C_{33}\psi$

where *X* , *Y* , *N,* are defined as in 1.1 above.

C_{ij} = coefficients of linear creep

ξ = longitudinal creep ratio = longitudinal creep speed/forward speed

α = lateral creep ratio = (lateral speed /forward speed) - steer angle

ψ = spin creep ratio = spin angular velocity/forward speed

2. Plane Motion in a Moving Coordinate Frame

$$\ddot{\mathbf{R}}_{\mathbf{O}_1} = (\dot{u} - v\Omega)\mathbf{i} + (\dot{v} + u\Omega)\mathbf{j}$$

$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ axis system fixed to body at point O_1

where

u = speed of point O_1 in $\mathbf{i}$ direction

v = speed of point O_1 in $\mathbf{j}$ direction

$\Omega\mathbf{k}$ = absolute angular velocity of body

3. Routh-Hurwitz stability criteria

$$\left(a_2 \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_0\right) y = x(t)$$

Stable if all $a_i > 0$

$$\left(a_3 \frac{d^3}{dt^3} + a_2 \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_0\right) y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 > a_0 a_3$

$$\left(a_4 \frac{d^4}{dt^4} + a_3 \frac{d^3}{dt^3} + a_2 \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_0\right) y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

DATA ON VEHICLE VIBRATION

Random Vibration

$$\mathrm{E}\left[x(t)^2\right] = \frac{1}{T}\int_{t=0}^{t=T} x^2(t)\,dt = \int_{\omega=-\infty}^{\omega=\infty} S_x(\omega)\,d\omega \qquad \left(\text{or } \int_{\omega=0}^{\omega=\infty} S_x(\omega)\,d\omega \quad \text{if } S_x(\omega) \text{ is single sided}\right)$$

$S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)$ (Spectrum of time derivative of x is ω^2 times spectrum of x).

Single Input – Single Output

$$S_y(\omega) = \left|H_{yx}(\omega)\right|^2 S_x(\omega)$$

$$y(\omega) = H_{yx}(\omega)x(\omega)$$

Two Input – Two Output

$$\begin{Bmatrix} y_1(\omega) \\ y_2(\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{Bmatrix} x_1(\omega) \\ x_2(\omega) \end{Bmatrix}$$

$$\begin{bmatrix} S_{11}^y(\omega) & S_{12}^y(\omega) \\ S_{21}^y(\omega) & S_{22}^y(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^* \begin{bmatrix} S_{11}^x(\omega) & S_{12}^x(\omega) \\ S_{21}^x(\omega) & S_{22}^x(\omega) \end{bmatrix} \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^{\mathrm{T}}$$

* means complex conjugate, T means transpose

If x_1 and x_2 are uncorrelated:

$$S_{(x_1+x_2)}(\omega) = S_{x_1}(\omega) + S_{x_2}(\omega)$$

$$S_{12}^x(\omega) = S_{21}^x(\omega) = 0$$

$$\mathrm{E}\left[(x_1(t) + x_2(t))^2\right] = \mathrm{E}\left[x_1(t)^2\right] + \mathrm{E}\left[x_2(t)^2\right]$$