Prof D Cebon

ENGINEERING TRIPOS PART IIB 2014 MODULE 4C8 SOLUTIONS.

Q1

(a) Assumptions used to derive standard "bicycle" model:

- Small angles
- Linear creep, neglect tyre realigning moments
- 2 tyres on each axle have same slip angle
- Neglect motion of sprung mass on suspension

(b) With F applied forward of G, $N = Fx$ and $Y = F$.
In the steady state, $\dot{v} = \dot{\Omega} = 0$, so the eq's become:

$$\begin{bmatrix} C/u & Cs/u + mu \\ Cs/u & Cq^2/u \end{bmatrix} \begin{Bmatrix} v_{ss} \\ \Omega_{ss} \end{Bmatrix} = \begin{Bmatrix} F \\ xF \end{Bmatrix} \quad \text{— (1)}$$

Where $C = C_f + C_r$, $s = \dfrac{aC_f - bC_r}{C_f + C_r}$, $q^2 = \dfrac{a^2 C_f + b^2 C_r}{C_f + C_r}$ — (2)

Inverting (1) gives

$$\begin{Bmatrix} v_{ss} \\ \Omega_{ss} \end{Bmatrix} = \frac{F \begin{bmatrix} Cq^2/u & -\left(\frac{Cs}{u} + mu\right) \\ -Cs/u & C/u \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}}{\left(C/u\right)\left(Cq^2/u\right) - \left(Cs/u\right)\left(Cs/u + mu\right)} \quad \text{— (3)}$$

Using (2), denominator is $\frac{1}{u^2}\left[C_f C_r l^2 - Csmu^2\right]$, $(l = a + b)$

So (3) $\rightarrow$

$$\left.\begin{aligned} \frac{v_{ss}}{u} &= \beta_{ss} = \left[\frac{Cq^2 - x(Cs + mu^2)}{C_f C_r l^2 - Csmu^2}\right] F \\ \& \quad \frac{\Omega_{ss}}{u} &= \left[\frac{C(x - s)}{C_f C_r l^2 - Csmu^2}\right] F \end{aligned}\right\} \quad \text{— (4)}$$

(c) If $x = s$ then $\Omega_{ss} = 0$ for any vehicle. This point on the vehicle $[x = (aC_f - bC_r)/(C_f + C_r)]$ is known as the Neutral Steer Point (NS).

1 (cont) The *static margin* is the distance that the NS point lies behind G, normalised by the wheelbase, l, ie $SM = -s/l$. The vehicle effectively rotates about NS. *Understeer* and *oversteer* refer to the motion of the vehicle when F is applied at G, ie $x = 0$.
In this case, from (4):

If $\dfrac{s=0}{SM=0}$ $\quad \dfrac{\Omega_{ss}}{u} = 0$ and $\beta_{ss} = \dfrac{cq^2}{C_f C_r l^2}$ = Const all speeds

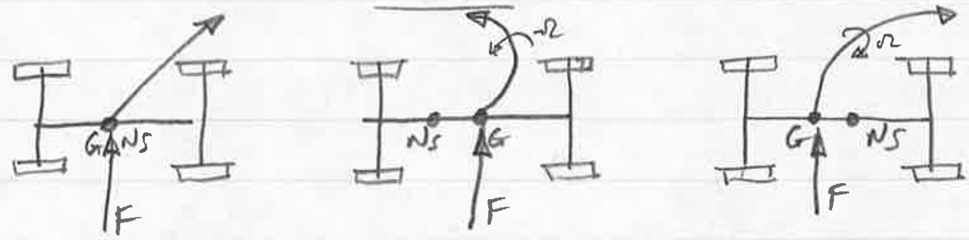
This is *neutral steer* $\Rightarrow$ NS coincides with G.

If $\dfrac{s<0}{SM>0}$ then $\dfrac{\Omega_{ss}}{u} > 0$ & $\beta > 0$ all speeds

This is *understeer* $\Rightarrow$ NS aft of G

If $\dfrac{s>0}{SM<0}$ then for $u < \sqrt{\dfrac{C_f C_r l^2}{CSM}}$, $\dfrac{\Omega_{ss}}{u} < 0$ & $\beta_{ss} > 0$

This is *oversteer* $\Rightarrow$ NS forward of G



Neutral steer ($SM=0$) | Understeer ($SM>0$) | Oversteer ($SM<0$)

(d) If δ is applied and held, in the steady state $\Omega = u/R$, $\dot{\Omega} = \dot{v} = 0$, Eq's of motion $\Rightarrow$

$$\begin{bmatrix} c & cs + mu^2 \\ cs & cq^2 \end{bmatrix} \begin{Bmatrix} \beta \\ 1/R \end{Bmatrix} = \begin{Bmatrix} C_f \delta \\ aC_f \delta \end{Bmatrix} \qquad \text{— (5)}$$

Solving (5) for R gives $\dfrac{1/R}{\delta} = \dfrac{cC_f(a-s)}{C_f C_r l^2 - CSmu^2} = \dfrac{l C_f C_r}{C_f C_r l^2 - CSmu^2}$

1(cont) So $\delta = \frac{l}{R}\left(1 - \frac{C_s m u^2}{l^2 C_f C_r}\right)$ ——⑥

Differentiate ⑥ to find speed sensitivity:

$$\frac{d\delta}{du} = -\frac{2l}{R}\left(\frac{C_s m}{l^2 C_f C_r}\right) \quad —— ⑦$$

Neutral steer $(s=0) \rightarrow \delta = l/R$ & $d\delta/du = 0 \quad \forall u$

Understeer $(s<0) \rightarrow \frac{d\delta}{du} > 0$, all speeds. So more steer angle is needed for higher speeds

Oversteer $(s>0) \rightarrow \frac{d\delta}{du} < 0$, all speeds. Vehicle becomes unstable with $\delta = 0$ when $u = \sqrt{\frac{C_f C_r l^2}{C_s m}}$

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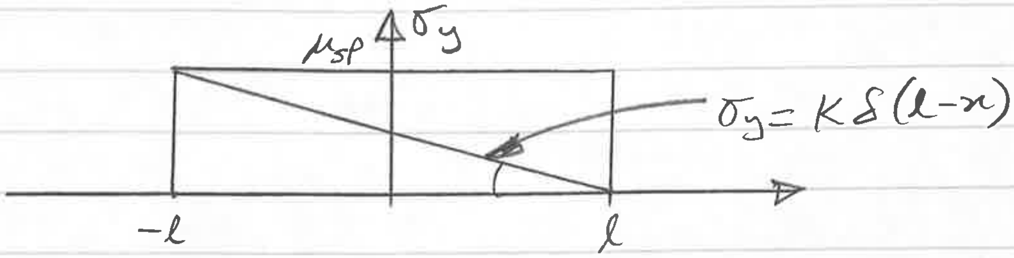
2.



(a) Microslip first occurs at the rear of the contact area when

$$\sigma_y(x=-l) = \mu_s p$$

$$\Rightarrow \quad 2Kl\delta_{crit} = \mu_s p \Rightarrow \delta_{crit} = \frac{\mu_s p}{2Kl} \qquad (1)$$

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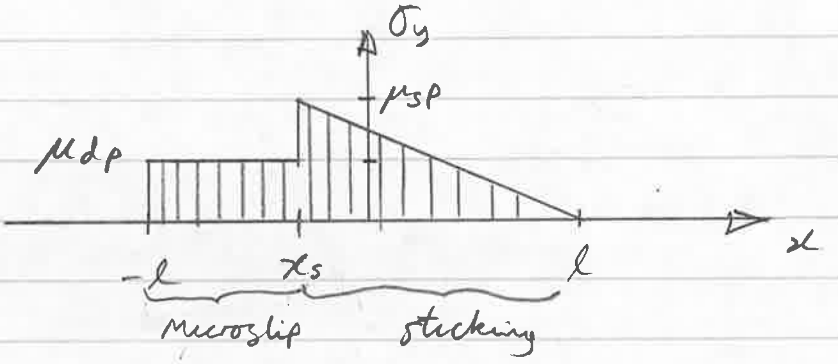
(b)



When $\delta > \delta_{crit}$, microslip first starts at x_s, where:

$$K\delta(l-x_s) = \mu_s p \Longrightarrow x_s = l - \frac{\mu_s p}{K\delta} \quad — (2)$$

The lateral force is given by area under $\sigma_y - x$

$$Y = \underbrace{2h\int_{-l}^{x_s} \mu_d p \, dx}_{\text{microslip}} + \underbrace{2h\int_{x_s}^{l} K\delta(l-x)\, dx}_{\text{no slip — linear region}}$$

$$= 2h\mu_d p(x_s+l) + 2hK\delta\left[-\tfrac{1}{2}(l-x)^2\right]_{x_s}^{l}$$

$$= 2h\mu_d p(x_s+l) + hK\delta(l-x_s)^2 \quad — (3)$$

Substituting x_s from (2) into (3) gives

$$Y = 2h\mu_d p\left(2l - \frac{\mu_s p}{k\delta}\right) + hk\delta\left(\frac{\mu_s p}{k\delta}\right)^2$$

$$= 4h\mu_d p l - \frac{2h\mu_d\mu_s p^2}{k\delta} + \frac{h\mu_s^2 p^2}{k\delta}$$

$$= 4h\mu_d p l + \frac{p^2 h}{k\delta}\mu_s(\mu_s - 2\mu_d) \qquad (4)$$

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(check if $\mu_s = \mu_d = \mu$ $Y = 4h\mu p l - \frac{p^2 h}{k\delta}\mu^2$ ✓ as per lecture notes)

(c) For $\delta > \delta_{crit}$, Y is given by eq. 4.

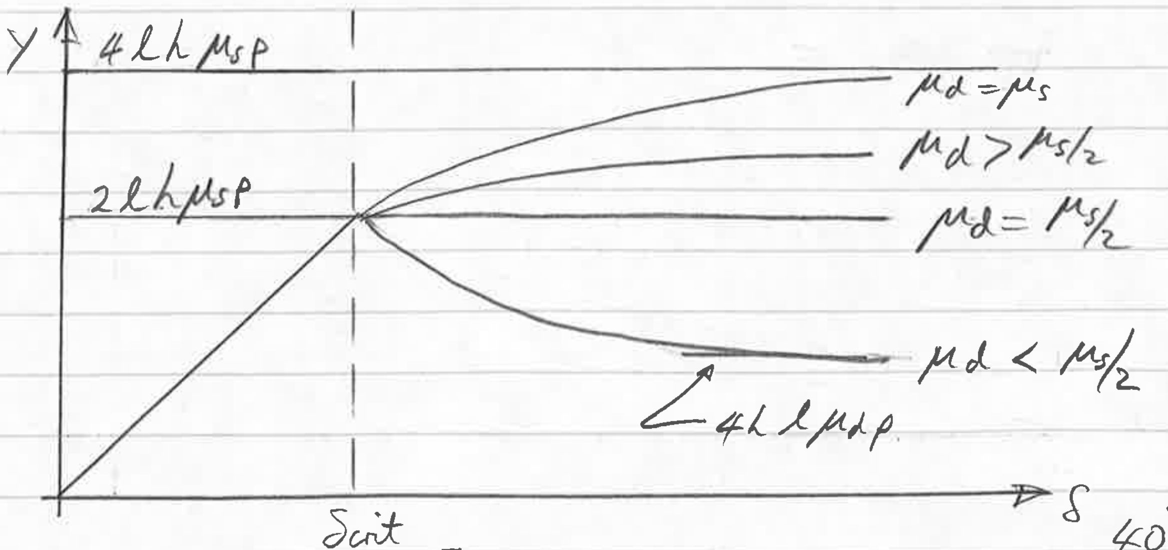
$$\Rightarrow \frac{dY}{d\delta} = -\frac{p^2 h}{k\delta^2}\mu_s(\mu_s - 2\mu_d)$$

$$\frac{dY}{d\delta} < 0 \quad \text{if} \quad \mu_s - 2\mu_d > 0$$

ie if $\mu_s > 2\mu_d \rightarrow \mu_d < \frac{\mu_s}{2}$

At $\delta = \delta_{crit}$, $x_s = -l$ & (3) $\Rightarrow Y = hk4l^2\delta_{crit}$
$= 2hl\mu_s p$

& at $\delta \rightarrow \infty$, (4) $\Rightarrow Y = 4hl\mu_d p$



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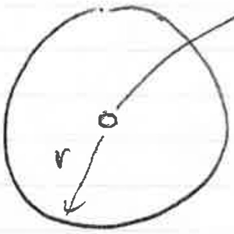
3. a) If $u = \frac{GM}{|\underline{r}|} = \frac{GM}{r}$

then $\underline{F} = \nabla u = \frac{\partial u}{\partial r}\underline{e}_r = -\frac{GM}{r^2}\underline{e}_r$

This is consistent with Newton's expression



Blob of matter vol. V_0, density ρ at centre of shell, radius r

Force (outwards) at surface of shell is $\underline{F} = -\frac{G\rho V_0}{r^2}$

Applying Gauss' theorem,

$$\iiint \nabla\cdot\underline{F}\, dV = \iint \underline{F}\cdot d\underline{s} = -\frac{G\rho V_0}{r^2}\cdot 4\pi r^2$$

$$= -4\pi G\rho V_0$$

Since this does not depend on r, we can shrink the shell just to contain V_0:

$$V_0\,\nabla\cdot\underline{F} = -4\pi G\rho V_0$$

$$\therefore\ \nabla\cdot\underline{F} = \nabla\cdot\nabla u = \nabla^2 u = -4\pi G\rho$$

b) Assuming $u = R(r)T(\theta)$, we can rewrite Laplace's equation as:

$$\nabla^2 u = \frac{T}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial T}{\partial\theta}\right] = 0$$

or $$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\frac{1}{T}\frac{d^2T}{d\theta^2} - \frac{\cot\theta}{T}\frac{dT}{d\theta}$$

3 (cont)

LHS and RHS must both be constant, for this equation to hold. Letting the constant be $n(n+1)$, we get

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

and $$\frac{d^2T}{d\theta^2} + \cot\theta \frac{dT}{d\theta} + n(n+1)T = 0$$

as required.

Let $R = Kr^p$

Then $r^2 p(p-1)r^{p-2} + 2rpr^{p-1} - n(n+1)r^p = 0$

i.e. $$p^2 - p + 2p - n(n+1) = 0$$

$$p^2 + p - n(n+1) = 0$$

$$(p-n)(p+n+1) = 0$$

So $\underline{R = K_1 r^n + K_2 r^{-(n+1)}}$

The D.E. for $T(\theta)$ can also be solved, using Legendre polynomials, and a family of solutions (one for each value of n) can be built up, as shown in the expression. The values of J_n are then set to model the earth's potential similar to fourier coefficients.

3 (cont)

(b) i) There is no need for an $n=1$ term if the origin is placed at the earth's centre of mass.

ii) The $n=2$ term models the earth's equatorial 'bulge'.

iii) The variations of U with ϕ are small, and tend to be smeared out because the earth is spinning beneath the satellite.

c) The higher terms cause secular effects - slow, long-term changes in the right ascension of the ascending node, and the argument of perigee.

They also cause fast acting (twice per orbit) effects, which change i, Θ and r from the values predicted by Keplerian orbits.

4a) Applying the expression for the earth's potential up to the J_2 term with $\cos\theta = 0$ gives

$$\mathcal{U}(r,\theta) = \mu/r\left[1 - \left(\frac{R}{r}\right)^2 J_2\left(-\tfrac{1}{2}\right) + \dots\right]$$

$$\approx \mu/r + \frac{\mu R^2 J_2}{2r^3}$$

Hence force per unit mass on satellite

$$\underline{F} = \frac{\partial \mathcal{U}}{\partial r}\underline{e}_r \approx -\left(\frac{\mu}{r^2} + \frac{3\mu R^2 J_2}{2r^4}\right)\underline{e}_r$$

i.e. a central force, acting inwards.

We can therefore use the formula at section 5 of the data sheet, i.e.

$$\frac{d^2u}{d\theta^2} + u = \frac{2\mu u^2 + 3\mu R^2 J_2 u^4}{2h^2u^2}$$

b) If $u = u_0 + x$, where u_0 = constant and $u_0 >> x$, then $\frac{d^2u}{d\theta^2} = \frac{d^2x}{d\theta^2}$, so

$$\frac{d^2x}{d\theta^2} = \frac{\mu}{h^2} - u_0 - x + \frac{3\mu R^2 J_2}{2h^2}(u_0^2 + 2u_0x + x^2)$$

The x^2 term can be ignored, giving

$$\frac{d^2x}{d\theta^2} = \left(\frac{3\mu R^2 J_2}{2h^2}\right)u_0^2 - u_0 + \frac{\mu}{h^2} - \left(1 - \frac{3\mu R^2 J_2}{h^2}u_0\right)x$$

4 (cont)

Now let $x = u_0 e \cos(\lambda\theta)$:

$$-u_0 e \lambda^2 \cos(\lambda\theta) = \left[\left(\frac{3\mu R^2 J_2}{2h^2}\right)u_0^2 - u_0 + \frac{\mu}{h^2}\right] - \left\{1 - \left(\frac{3\mu R^2 J_2}{h^2}\right)u_0\right\} u_0 e \cos(\lambda\theta)$$

For this to work, $[\text{—}] = 0$, and the other two terms must equate - dividing both sides by $u_0 e \cos(\lambda\theta)$ gives:

$$\lambda^2 = 1 - \frac{3\mu R^2 J_2}{h^2} u_0$$

Since $u_0 \approx \mu / h^2$

$$\lambda = \sqrt{1 - 3R^2 J_2 u_0^2}$$

If u_0 is, say, $\frac{1}{nR}$ where $n > 1$,

we get $\lambda = \sqrt{1 - \frac{3J_2}{n^2}}$ which is just less than 1, as $J_2 << 1$

This means that the perigee of the orbit (where $\cos(\lambda\theta) = 1$) slowly rotates round the earth.

c) If $n = 2$, $\lambda = \sqrt{1 - \frac{3J_2}{4}} \approx 1 - 4.06 \times 10^{-4}$

So argument of perigee changes by 1 revolution in $\frac{1}{4.06 \times 10^{-4}} = \underline{2,463}$ orbits