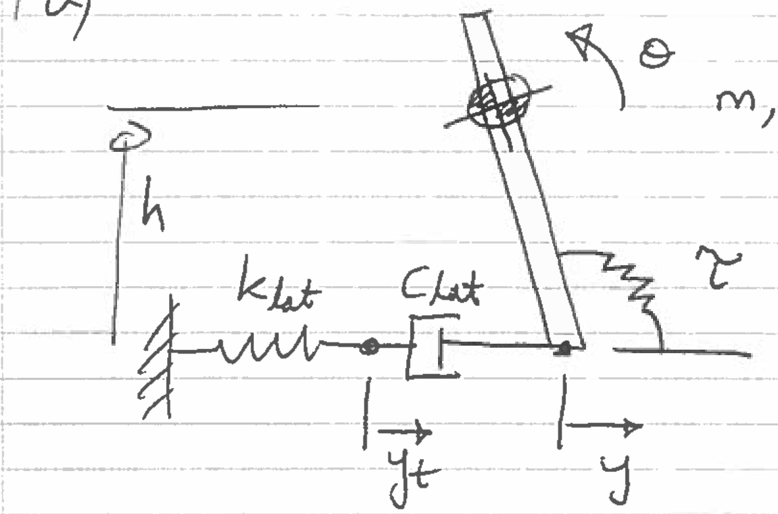
1 a)



$m = m_s + m_u = 800\,kg + 90\,kg = \underline{890\,kg}$

$\tau = 2 k_t T^2 = 2 \cdot 200 \cdot 10^3 \cdot 0.75^2 = \underline{225\ kNm/rad}$

$k_{lat} = \underline{178\ kN/m}$, c_{lat} as in question.

to find h : $m_s \cdot h_s + m_u \cdot h_u = m \cdot h$

$\therefore h = \dfrac{800 \cdot 0.5 + 90 \cdot 0.3}{890}$

$\underline{h = 0.48m}$

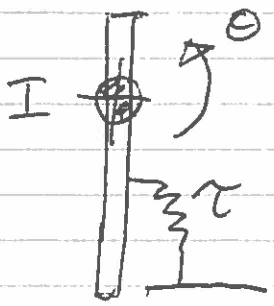
to find I :

$m_s h_s^2 + I_s + m_u h_s^2 + I_u = m h^2 + I$

$I = 800 \cdot 0.5^2 + 460 + 90 \cdot 0.3^2 + 51 - 890 \cdot 0.48^2$

$\underline{I = 514\ kgm^2}$

b) i) high speed $c_{lat} \to 0$



no lateral forces $\therefore$ consider only roll motion. Newton's 2nd Law:

$\ddot{\theta} I = -\theta \tau$

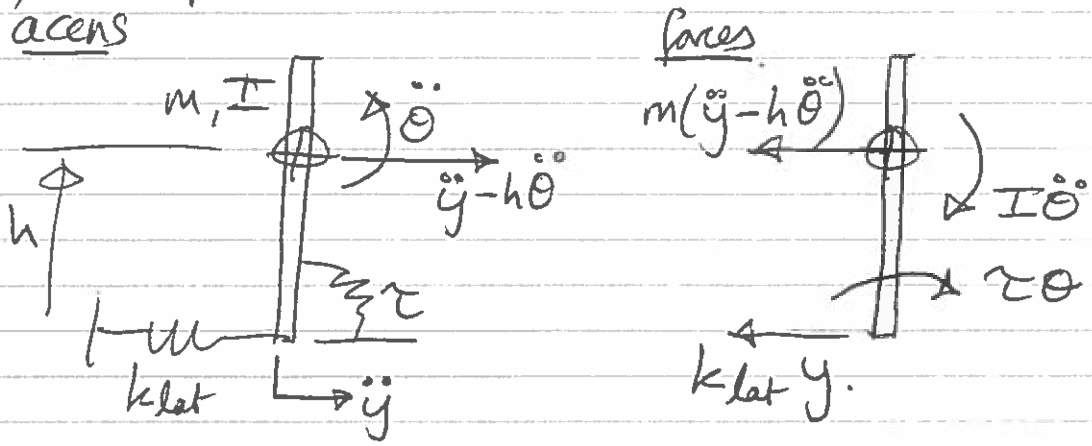
$\ddot{\theta} I + \theta \tau = 0$

$\therefore \omega_n = \sqrt{\dfrac{\tau}{I}} = \sqrt{\dfrac{225 \cdot 10^3}{514}} = 20.9\ rad/s$

$\underline{= 3.33\ Hz}$

also lateral rigid body mode at $\underline{0\ Hz}$

1 ii) low speed $C_{lat} \to \infty$



mts about CoM $\quad I\ddot{\theta} + k_{lat}\,y\,h + \tau\theta = 0$

sum horiz forces to zero $\quad -k_{lat}\,y - m(\ddot{y} - h\ddot{\theta}) = 0$

matrix: $\begin{bmatrix} I & 0 \\ -mh & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} \tau & k_{lat}h \\ 0 & k_{lat} \end{bmatrix} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Laplace and $s = j\omega$:

$$\left\{ -\omega^2 \begin{bmatrix} I & 0 \\ -mh & m \end{bmatrix} + \begin{bmatrix} \tau & k_{lat}h \\ 0 & k_{lat} \end{bmatrix} \right\} \begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Solutions when $\det \begin{bmatrix} \tau - \omega^2 I & k_{lat}h \\ \omega^2 mh & k_{lat} - m\omega^2 \end{bmatrix} = 0$

$$\therefore (\tau - \omega^2 I)(k_{lat} - m\omega^2) - k_{lat}h^2\omega^2 m = 0$$
$$\tau k_{lat} - \tau m\omega^2 - \omega^2 Ik + mI\omega^4 - kh^2\omega^2 m = 0$$
$$\omega^4 mI + \omega^2(-\tau m - Ik - kh^2 m) + \tau k = 0$$

substitute numerical values to give

$$\omega^4\, 457.5 . 10^3 - \omega^2\, 328.2 . 10^6 + 4.01 . 10^{10} = 0$$

quadratic formula

$$\omega^2 = \frac{328.2\,10^6 \pm \sqrt{(328.2 . 10^6)^2 - 4 . 457.5 . 10^3 . 4.01 . 10^{10}}}{2 . \ 457.5 . 10^3}$$

$$\omega^2 = 358 \pm 203 \ \cancel{\text{rad}}$$

$$\therefore \ f = \frac{\omega}{2\pi} = \underline{\underline{1.98\,\text{Hz and } 3.76\,\text{Hz}}}$$

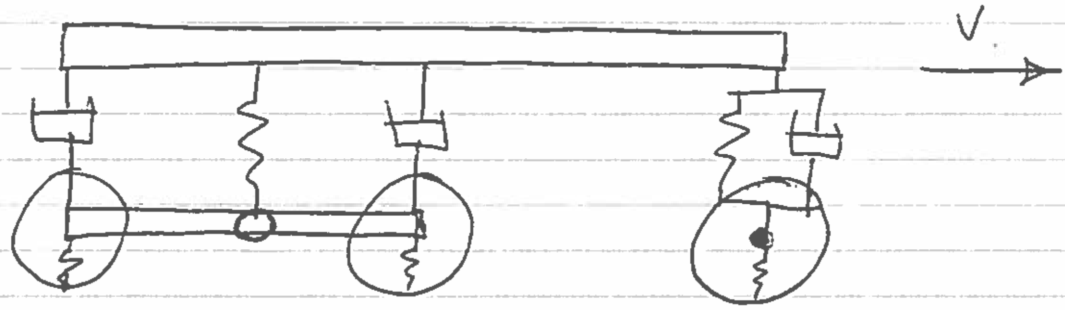
1.c) ~~At~~ high speed, $V = 30 m/s$:

High speed means small excitation at the natural frequency of the 3.33 Hz lateral-roll mode. However this mode has negligible damping, leading to large amplitude resonance at 3.33 Hz in $\ddot{\theta}_s$ and $\ddot{z}_p$.

At low speed, $V = 1 m/s$:

Low speed means significant excitation at the natural frequencies of the 1.98 Hz and 3.76 Hz lateral-roll modes. Both modes have negligible damping leading to large magnitude resonances at both frequencies in $\ddot{\theta}_s$ and $\ddot{z}_p$. Again, the unsprung mass roll mode is not present.

2 a)



- conventional parallel spring-damper on front axle
- spring at centre of beam to ensure equal static forces on middle and rear axles.
- dampers at each end of beam to ensure unsprung mass modes are damped.

b) vertical displacement at centre of beam $z' = \frac{z_2 + z_3}{2}$

vertical displacement of body $z = \frac{z_1}{3} + z'\frac{2}{3}$

hence $z = \frac{z_1}{3} + (z_2 + z_3)\frac{1}{3}$

consider time delays between axles

$$z_2(t) = z_1\left(t - \frac{a}{V}\right) \quad , \quad z_3(t) = z_1\left(t - \frac{2a}{V}\right)$$

$$\therefore \quad z(t) = \frac{1}{3}\left(z_1(t) + z_1\left(t - \frac{a}{V}\right) + z_1\left(t - \frac{2a}{V}\right)\right)$$

Laplace transform and $s = j\omega$

$$3 \cdot \frac{z(j\omega)}{z_1(j\omega)} = 1 + e^{-j\omega\frac{a}{V}} + e^{-j\omega\frac{2a}{V}}$$

De Moivre $= 1 + \cos\frac{\omega a}{V} - j\sin\frac{\omega a}{V} + \cos\frac{2\omega a}{V} - j\sin\frac{2\omega a}{V}$

let $\frac{\omega a}{V} = \Omega$. $= 1 + \cos\Omega + \cos 2\Omega - j(\sin\Omega + \sin 2\Omega)$

but $\cos\Omega + \cos 2\Omega = 2\cos\frac{3\Omega}{2}\cos\frac{\Omega}{2}$

and $\sin\Omega + \sin 2\Omega = 2\sin\frac{3\Omega}{2}\cos\frac{\Omega}{2}$

$$\therefore \quad 3\left|\frac{z(j\omega)}{z_1(j\omega)}\right| = \sqrt{\left(1 + 2\cos\frac{3\Omega}{2}\cos\frac{\Omega}{2}\right)^2 + \left(2\sin\frac{3\Omega}{2}\cos\frac{\Omega}{2}\right)^2}$$

$$= \sqrt{1 + 4\cos\frac{3\Omega}{2}\cos\frac{\Omega}{2} + 4\cos^2\frac{3\Omega}{2}\cos^2\frac{\Omega}{2} + 4\sin^2\frac{3\Omega}{2}\cos^2\frac{\Omega}{2}}$$

$$= \sqrt{1 + 4\cos\frac{\Omega}{2}\left(\cos\frac{3\Omega}{2} + \cos\frac{\Omega}{2}\right)}$$

$$= \sqrt{1 + 4\cos\frac{\Omega}{2}\, 2\cos\Omega\cos\frac{\Omega}{2}}.$$

$$\left|\frac{Z(j\omega)}{Z_1(j\omega)}\right| = \sqrt{\frac{1 + 8\cos^2\frac{\Omega}{2}\cos\Omega}{9}}$$

where $\Omega = \frac{\omega a}{V}$

c) whole number of cycles between axles when $\Omega = \frac{\omega a}{V} = n2\pi$, n is integer.

by inspection $\left|\frac{Z(j\omega)}{Z_1(j\omega)}\right| = 1$ when $\cos^2\frac{\Omega}{2}\cos\Omega = +1$

check $\cos^2\frac{\Omega}{2} = \cos^2(n\pi) = +1$ for all n.

$\cos\Omega = \cos(n2\pi) = +1$ for all n.

Hence gain is 1 when whole number of cycles are between adjacent axles.

d) by inspection $\left|\frac{Z(j\omega)}{Z_1(j\omega)}\right| = 0$ when $\cos^2\left(\frac{\Omega}{2}\right)\cos(\Omega) = -\frac{1}{8}$

$$\frac{1}{2}(1+\cos\Omega)\cos\Omega = -\frac{1}{8}.$$

$$\cos^2\Omega + \cos\Omega + \frac{1}{4} = 0$$

$$\cos\Omega = \frac{-1 \pm \sqrt{1 - 4.1.\frac{1}{4}}}{2}.$$

$$\cos\Omega = -\frac{1}{2}$$

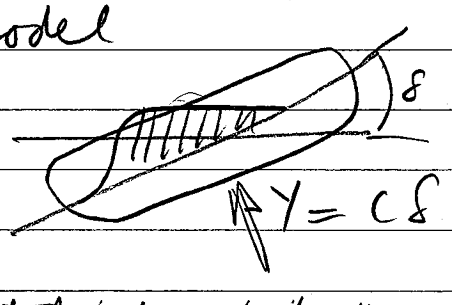
$$\therefore \Omega = \frac{\omega a}{V} = \frac{2\pi}{3}, \frac{4\pi}{3} \quad (+\, n2\pi)$$

Q3

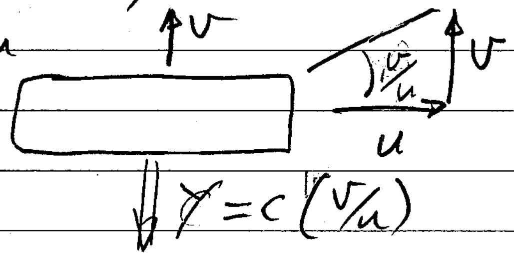
(a) Longitudinal creep forces occur in curving of a bogie because the wheels are coned and run at different speeds as the bogie yaws & moves laterally. The longitudinal forces generate a yaw moment about the centre of the bogie and this moment enters the equations of motion. In steady curving the forces on left & right wheels are equal & opposite & so have no net braking or accelerating effect.

Cars either have differentials or free wheels which mean they don't generate any longitudinal forces during cornering at constant speed.

(b) Yaw angle generates creep forces Y as explained by brush model



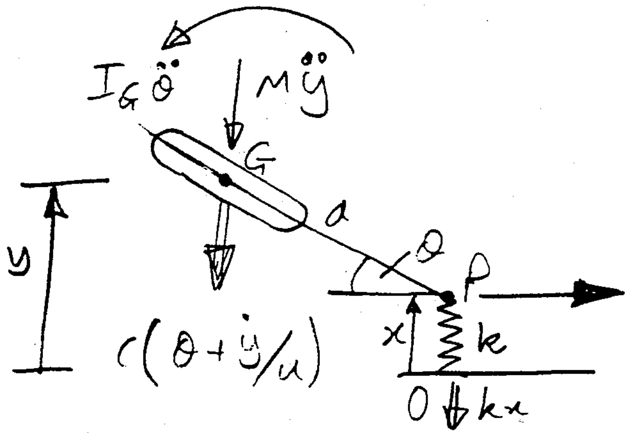
When subjected to a lateral speed v & forward speed u, the effective yaw angle is v/u



So in combined motion $Y = c(v/u - \delta)$

Q3cont

(bi)



$y = x + a\theta$

$\dot{y} = \dot{x} + a\dot{\theta}$

$\ddot{y} = \ddot{x} + a\ddot{\theta}$

$\downarrow \Sigma F: \quad c\left(\theta + \left(\frac{\dot{x} + a\dot{\theta}}{u}\right)\right) + kx + m\left(\ddot{x} + a\ddot{\theta}\right) =$

$\circlearrowleft \Sigma M_G: \quad I_G\ddot{\theta} - kxa = 0$

(bii) ie $\begin{bmatrix} m & ma \\ 0 & I_G \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c/u & \frac{ca}{u} \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & c \\ -ka & 0 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix}$

put $\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{st}$.

$$\begin{vmatrix} ms^2 + cs/u + k & mas^2 + \frac{ca}{u}s + c \\ -ka & I_G s^2 \end{vmatrix} = 0$$

$$I_G s^2\left(ms^2 + cs/u + k\right) + \left(mas^2 + \frac{ca}{u}s + c\right)ka = 0$$

$$\underbrace{mI_G}_{a_4} s^4 + \underbrace{\frac{I_G c}{u}}_{a_3} s^3 + \underbrace{\left(I_G + ma^2\right)k}_{a_2} s^2 + \underbrace{\frac{kca^2}{u}}_{a_1} s + \underbrace{ck}_{a_0}$$

Routh Hurwitz:

$a_1 a_2 a_3 > a_1^2 a_4 + a_3^2 a_0$ & all a's > 0

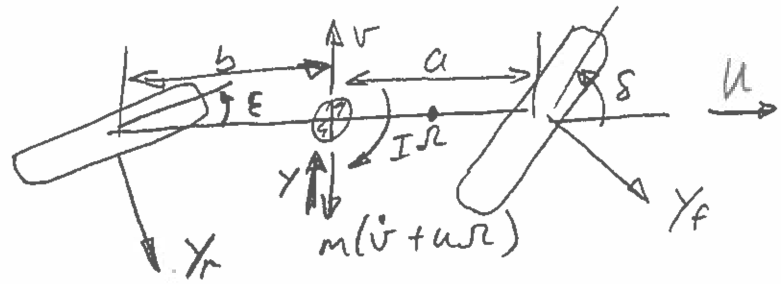
$$\cancel{\frac{kca^2}{u}}\left(I_G + ma^2\right)k\,\cancel{\frac{I_G c}{u}} > \frac{k^2c^2a^4}{u^2} m I_G + \frac{I_G^2 c^2}{u^2} kc$$

$ak\left(I_G + ma^2\right) > I_G c + ma^3k$

$\cancel{I_G}ak + \cancel{ma^3k} > \cancel{I_G}c \cancel{+ma^3k}$

$k > c/a$ //

4. (a)



The force generated by the rear tyre is

$$Y_r = C_r \alpha_r = C\left(\frac{v - b\Omega}{u} - \varepsilon\right) \qquad \text{— ①}$$

but $\varepsilon = \gamma Y_r$ so $Y_r(1+\gamma C) = C\left(\frac{v-b\Omega}{u}\right)$

$$\Rightarrow Y_r = \frac{C}{u}\frac{(v - b\Omega)}{(1+\gamma C)} \qquad \text{— ②}$$

& $Y_f = C_f \alpha_f = C\left(\frac{v + a\Omega}{u} - \delta\right)$, as usual — ③

(b) Considering δ & ε to be small, lateral motion is

$$m(\dot{v} + u\Omega) + \frac{C}{u}\left(\frac{v - b\Omega}{1+\gamma C}\right) + C\left(\frac{v + a\Omega}{u} - \delta\right) - Y = 0 \qquad ④$$

& yaw motion is

$$I\dot{\Omega} + aC\left(\frac{v + a\Omega}{u} - \delta\right) - \frac{bC}{u}\left(\frac{v - b\Omega}{1+\gamma C}\right) = 0 \qquad \text{— ⑤}$$

In matrix form this is

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \dot{v} \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} \frac{C}{u}\left(\frac{1}{1+\gamma C} + 1\right) & mu + \frac{C}{u}\left(a - \frac{b}{1+\gamma C}\right) \\ \frac{C}{u}\left(a - \frac{b}{1+\gamma C}\right) & \frac{C}{u}\left(a^2 + \frac{b^2}{1+\gamma C}\right) \end{bmatrix}\begin{Bmatrix} v \\ \Omega \end{Bmatrix} = \begin{Bmatrix} Y + C\delta \\ aC\delta \end{Bmatrix} \qquad ⑥$$

$$\Rightarrow [M]\,\dot{\underline{y}} + [K]\,\underline{y} = \underline{F}$$

(c) (i) For stability calculation, put $\underline{F} = 0$ and let $\underline{y} = \underline{Y}e^{\lambda t} \Rightarrow \dot{\underline{y}} = \lambda \underline{Y} e^{\lambda t}$

Then ⑥ becomes $\left([K] + \lambda[M]\right)\underline{Y} = 0$

for which the stability condition is $\left|[K] + \lambda[M]\right| = 0$

Giving $a_2\lambda^2 + a_1\lambda + a_0 = 0$, for which the

-1- stability condition is that $a_0 > 0, a_1 > 0$ & $a_2 > 0$.

4 Cont.

(iii) For a constant steer angle Δ, the steady motion will be circular with $\dot{v} = \dot{\Omega} = 0$

So $F = C\Delta \begin{Bmatrix} 1 \\ a \end{Bmatrix}$ and ⑥ can be written

$$[K]\, \underline{y}_{ss} = \underline{F}$$

Invert to give $\begin{Bmatrix} V_{ss} \\ \Omega_{ss} \end{Bmatrix} = [K]^{-1} C\Delta \begin{Bmatrix} 1 \\ a \end{Bmatrix}$

solve for Ω_{ss} then put $\Omega = \frac{U}{R}$ for circular motion

(iv) for $\delta = \Delta e^{i\omega t}$, the response will be $\underline{y} = \begin{Bmatrix} \bar{V} \\ \bar{\Omega} \end{Bmatrix} e^{i\omega t}$

So ⑥ becomes $\left(i\omega[M] + [K]\right) \underline{Y} \cancel{e^{i\omega t}} = C\Delta \begin{Bmatrix} 1 \\ a \end{Bmatrix} \cancel{e^{i\omega t}}$

Invert: $\underline{Y} = \begin{Bmatrix} \bar{V} \\ \bar{\Omega} \end{Bmatrix} = \left([K] + i\omega[M]\right)^{-1} C\Delta \begin{Bmatrix} 1 \\ a \end{Bmatrix}$

(d) When subjected to a steady side force, Y, the steady state motion will be $\dot{v} = \dot{\Omega} = 0$
So ⑥ can be written

$$[K]\, \underline{y}_{ss} = \underline{F} \quad , \text{where } \underline{F} = \begin{Bmatrix} Y \\ 0 \end{Bmatrix}$$

Invert this to give $\begin{Bmatrix} V_{ss} \\ \Omega_{ss} \end{Bmatrix} = [K]^{-1} \begin{Bmatrix} Y \\ 0 \end{Bmatrix}$

$$\text{So } \begin{Bmatrix} V_{ss} \\ \Omega_{ss} \end{Bmatrix} = \frac{\begin{bmatrix} C/u\left(a^2 + \frac{b^2}{1+\gamma c}\right) & -\left[Mu + C/u\left(a - \frac{b}{1+\gamma c}\right)\right] \\ -C/u\left(a - \frac{b}{1+\gamma c}\right) & C/u\left(a - b/_{1+\gamma c}\right) \end{bmatrix} \begin{Bmatrix} Y \\ 0 \end{Bmatrix}}{\underbrace{\left[C/u\left(\frac{1}{1+\gamma c} + 1\right)\right]\left(C/u\left(a^2 + \frac{b^2}{1+\gamma c}\right)\right) - \left[\left(Mu + C/u\left(a - \frac{b}{1+\gamma c}\right)\right)\left(C/u\left(a - \frac{b}{1+\gamma c}\right)\right)\right]}_{\text{'}\Delta\text{'}}}$$

$$\Omega_{ss} = \frac{-C/u\left(a - \frac{b}{1+\gamma c}\right) Y}{\Delta}$$

For Neutral Steer, $\Omega_{ss} = 0$ ie $a - \frac{b}{1+\gamma c} = 0$

$$\Rightarrow \gamma = \frac{b-a}{aC}$$

If $a = b$ then $\gamma = 0$ <u>as expected</u>

Question 1
Roll-plane vibration: Remarkably unpopular question! One very good answer, one fair, one hopeless.

Question 2
Wheelbase filtering. Part (a) was mixed: some quite good answers, some hopeless. Part (b) was generally well done, though some made heavy going of the algebra. Parts (c) and (d) were generally fine.

Question 3
Wheel Shimmy. A surprising number of students didn't remember how to take moments of d'Alembert forces and torques from second year… Do I use the parallel axis theorem or not? Surely they should leave university with this level of knowledge in dynamics?

Question 4
Four wheel steer: Part (a) was straightforward and well done. Part (b) was bookwork – yet quite a few students managed to ignore the 'u ' acceleration term due to the rotating coordinate frame. Part (c) was generally well done. Part (d) was fine. The overall average was high.