Version DC/2

EGT3
ENGINEERING TRIPOS PART IIB

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Monday 28 April 2014 2 - 3:30

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**Module 4C8**

APPLICATIONS OF DYNAMICS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided script paper

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Attachment: 4C8 datasheet, 2014 (5 pages)
Engineering Data Book

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.**

1 A 'bicycle' model of a car, with freedom to sideslip with velocity $v$ and yaw at rate $\Omega$, is shown in Fig. 1. The car moves at steady forward speed $u$ on a horizontal surface. It has mass $m$, yaw moment of inertia $I$, and lateral creep coefficients of $C_f$ and $C_r$ at the front and rear tyres. The lengths $a$ and $b$ and the steering angle $\delta$ are defined in the figure. A yawing moment $N$ acts on the vehicle and a lateral force $Y$ is applied at the centre of gravity G. The equations of motion in a coordinate frame rotating with the vehicle are given by:

$$m(\dot{v}+u\Omega)+(C_f+C_r)\frac{v}{u}+(aC_f-bC_r)\frac{\Omega}{u}=C_f\delta+Y$$

$$I\dot{\Omega}+(aC_f-bC_r)\frac{v}{u}+(a^2C_f+b^2C_r)\frac{\Omega}{u}=aC_f\delta+N$$

(a) State the assumptions needed to derive these equations of motion. [10%]

(b) Use the equations of motion to derive expressions for the steady state yaw rate and sideslip responses of the vehicle to a side force $F$ applied a distance $x$ forward of G, with $\delta = 0$. [30%]

(c) Define the terms *neutral steer point, static margin*, *oversteer* and *understeer*. Explain how the handling response of the car to a steady side force varies with the static margin and the forward speed. Sketch the paths of motion of a car with positive, negative and zero static margins. [30%]

(d) Derive an expression for the steering angle $\delta$ needed to turn the vehicle on a steady circular curve of radius $R$ with $Y = N = 0$. Use this expression to explain the terms oversteer and understeer for a steadily turning car. [30%]

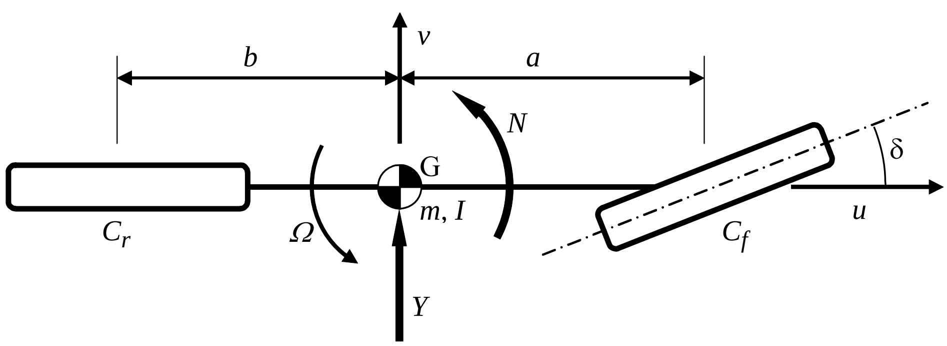


Fig. 1

2 A model of a tyre contact patch has length $2l$ and width $2h$. The contact pressure $p$ is uniform over the area of the patch. The tyre is rolled on a flat horizontal surface at yaw angle $\delta$ to the direction of motion. In order to improve the 'brush' model, it is suggested that different values are used for the static coefficient of friction $\mu_s$ and the dynamic coefficient of friction $\mu_d$, where $\mu_s > \mu_d$. The wheel is running freely (i.e. with no longitudinal creep) and the lateral brush stiffness per unit area is $K$.

(a) Show that the yaw angle when microslip first occurs at the rear of the contact patch is given by:

$$\delta_{crit} = \frac{\mu_s p}{2lK}$$ [10%]

(b) When the yaw angle increases above this value a region of microslip appears, in which the coefficient of friction is $\mu_d$. Show that the lateral force $Y$ generated by the tyre is given by:

$$Y = 4\mu_d plh + \frac{p^2 h}{K\delta}\mu_s\left(\mu_s - 2\mu_d\right)$$ [50%]

(c) Show that if $\mu_d < \mu_s/2$ the lateral force decreases with increasing yaw angle for $\delta > \delta_{crit}$. Sketch graphs of $Y$ against $\delta$ for $\mu_d < \mu_s/2$, $\mu_d = \mu_s/2$ and $\mu_d > \mu_s/2$. [40%]

3 (a) The gravitational potential at a position $\mathbf{r}$ due to a point mass $M$ at the origin is given by:

$$U(\mathbf{r}) = \frac{GM}{|\mathbf{r}|}, \text{ where } G \text{ is the universal gravitational constant.}$$

Show that this formula is consistent with Newton's formula for the gravitational force between two bodies, and use your result to prove that a body with distributed mass having density $\rho(\mathbf{r})$ will satisfy Poisson's equation, namely

$$\nabla^2 U = -4\pi G\rho$$ [20%]

(b) Laplace's equation in spherical co-ordinates can be written as

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial U}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0$$

By considering a solution in the form $U = R(r)T(\theta)$, show that solutions to Laplace's equation can be found by solving:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \quad \text{and}$$

$$\frac{d^2T}{d\theta^2} + \cot\theta\frac{dT}{d\theta} + n(n+1)T = 0$$

where $n$ is a constant. What are the solutions for $r$, for integer values of $n$?

Without going into further mathematical detail, explain how these equations lead to the approximate expression for the external potential of the earth

$$U(r,\theta) = \frac{\mu}{r}\left\{1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right\}$$

as defined on the data sheet. Give brief explanations for:

(i) the lack of a term involving $n = 1$ in the expression above;

(ii) the feature of the earth which is modelled by the $n = 2$ term;

(iii) the validity of assuming that $U$ is not a function of $\phi$. [60%]

(c) GPS satellites orbit the earth in near-circular orbits with an inclination of 55°. Describe two ways in which the $n = 2$ and higher terms in the expression above cause the Keplerian orbital parameters of GPS satellites to vary over time, stating which parameters are altered by each effect. [20%]

4 (a) A satellite orbits the earth in a near-circular orbit lying in the earth's equatorial plane. Using terms up to and including the $J_2$ term for the gravitational potential of the earth, find an expression for the force per unit mass acting on the satellite, as a function of its orbital radius $r$. Use this expression to write down a differential equation which gives the shape of the orbit, in terms of $u$ ( $= 1/r$) and $\theta$, the true anomaly. [30%]

(b) By making the substitution $u = u_0 + x$ (where $u_0$ is constant and much larger than $x$) derive a linearised differential equation relating $x$ to $\theta$, and show that this can be satisfied by a solution of the form $x = u_0 e \cos(\lambda\theta)$, where $e$ is the eccentricity. Use the simplifying approximation $u_0 \approx \mu/h^2$ to find an expression for $\lambda$ in terms of $R$, $J_2$ and $u_0$, where $R$ is the radius of the earth. What does this expression tell us about the orbit of the satellite? [50%]

(c) If the mean radius of the satellite's orbit is twice that of the earth, find the number of orbits which are needed for the argument of perigee to change by one complete revolution. [20%]

**END OF PAPER**