EGT3
ENGINEERING TRIPOS PART IIB

Monday 29 April 2024   2 to 3.40

Module 4C9

CONTINUUM MECHANICS

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C9 datasheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A polymer water tank is a thin-walled open-ended circular cylinder of radius R and wall thickness w. The tank is filled with water (density ρ) to a height H, as shown in cross-section in Fig. 1. The hydrostatic pressure

$$p = \rho g(H - z),$$

where g is the acceleration due to gravity, acts on the wall of the cylinder at a height z from the base. Within the tank wall, let directions 1, 2 and 3 correspond to hoop, longitudinal and through-thickness, respectively. Infinitesimal deformations can be assumed.

(a) The tank wall is modelled as an isotropic linear elastic solid with Young's modulus E and Poisson's ratio ν. The constitutive equations are

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

(i) Neglecting end effects and self-weight, explain why it can be assumed that the only non-zero stress component in the tank sidewall is the hoop stress $\sigma_{11}(z)$. Hence, write down an expression for the total elastic strain energy in the tank sidewall as a function of the hoop strain $\varepsilon_{11}(z)$. The base of the tank can be neglected. [15%]

(ii) Using the method of minimum potential energy, derive an expression for the hoop strain $\varepsilon_{11}(z)$ in the tank sidewall. [25%]

(b) To capture time dependent deformation of the polymer tank, a linear viscoelastic constitutive model is now used. Assuming Poisson's ratio to be time-independent, the 3D viscoelastic constitutive equations are

$$\varepsilon_{ij}(t) = (1+\nu)\int_0^t J_c(t-\tau)\frac{\partial\sigma_{ij}(\tau)}{\partial\tau}\,d\tau - \nu\int_0^t J_c(t-\tau)\frac{\partial\sigma_{kk}(\tau)}{\partial\tau}\delta_{ij}\,d\tau,$$

where $J_c(t)$ is the creep compliance.

(i) The uniaxial response of the material is given by the relationship

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta},$$

where E and η are material constants, and $\dot{\varepsilon}$ and $\dot{\sigma}$ are the strain and stress rates, respectively. Derive expressions for the relaxation modulus $E_r(t)$ and the creep compliance $J_c(t)$. [20%]

(ii) The tank is filled with water at a constant rate $\dot{H}$, such that $H = \dot{H}t$. Derive expressions for the rate of change of the tank radius, $\dot{R}$, and the rate of change of the wall thickness, $\dot{w}$, at a particular height z. End effects and the influence of the base should be neglected. [40%]

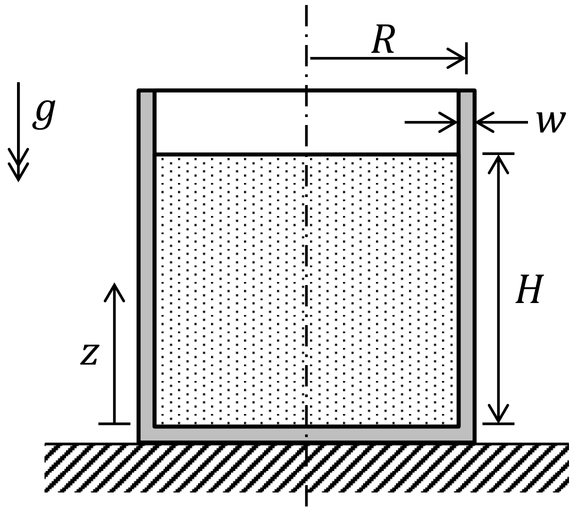

Fig. 1

2 (a) A body has volume V and surface S with outward unit normal vector $\mathbf{n}$, and undergoes infinitesimal deformations subject to external tractions $\mathbf{t}^e$ and body forces $\mathbf{b}$. Using index notation, prove the following.

(i) The principle of virtual work

$$\int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV - \int_S \mathbf{t}^e \cdot \delta\mathbf{u}\, dS - \int_V \mathbf{b} \cdot \delta\mathbf{u}\, dV = 0$$

is equivalent to the equilibrium relationships $\sigma_{ij,j} + b_i = 0$ and $t_i^e = \sigma_{ij} n_j$. [25%]

(ii) The balance of moments on the body

$$\int_S \mathbf{x} \times \mathbf{t}^e\, dS + \int_V \mathbf{x} \times \mathbf{b}\, dV = \mathbf{0},$$

where $\mathbf{x}$ is the position vector, requires symmetry of the stress tensor, $\sigma_{ij} = \sigma_{ji}$. [25%]

(b) Consider the decomposition of a second-order tensor $\mathbf{B}$ such that $\mathbf{B} = \mathbf{RU} = \mathbf{VR}$, where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive-semidefinite.

(i) Prove that such a decomposition exists for all tensors $\mathbf{B}$. Prove that, when $\det\mathbf{B} > 0$, $\mathbf{R}$ is a proper orthogonal tensor, and $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive-definite tensors. [25%]

(ii) Show the significance of this decomposition for defining strain measures in terms of the deformation gradient $\mathbf{F}$. [10%]

(iii) Given that $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, for the decompositions $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ provide geometric interpretations of $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$, supported by sketches. [15%]

3 (a) (i) The deformation of a body is given by

$$\boldsymbol{\phi}(\mathbf{X}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} pX_1 + a \\ qX_2 + b \\ rX_3 + c \end{bmatrix},$$

where p, q, r, a, b and c are constants. Calculate the deformation gradient $\mathbf{F}$ and give the conditions on the constants such that the deformation is admissible. [10%]

(ii) Consider a unit cube that deforms according to

$$x_1 = X_1 + \beta t^2 X_2, \quad x_2 = X_2, \quad x_3 = X_3 + 3X_2,$$

where β is a parameter and t is time. Calculate the deformation gradient and prove that the volume of the cube is preserved. [10%]

(b) Consider the motion of a body given by

$$\boldsymbol{\phi}(\mathbf{X}, t) = \mathbf{R}(t)\mathbf{X} + \mathbf{c}(t),$$

where $\mathbf{R}(t)$ is a rotation tensor.

(i) What type of motion does this expression describe? [10%]

(ii) Give an expression for the inverse motion, i.e. an expression for $\mathbf{X}$ in terms of the spatial coordinates $\mathbf{x}$. [10%]

(iii) Give expressions for the material velocity field $\mathbf{V}(\mathbf{X}, t)$ and for the spatial velocity field $\mathbf{v}(\mathbf{x}, t)$. [20%]

(c) The first Piola–Kirchhoff stress $\mathbf{P}$ is work-conjugate to the deformation gradient $\mathbf{F}$, i.e. $\mathcal{P} = \mathbf{P} : \dot{\mathbf{F}}$, where $\mathcal{P}$ is the stress power (per unit reference volume).

(i) For a body with a pre-stress ($\mathbf{P} \neq \mathbf{0}$), prove that the stress power is zero for a time-varying rigid body motion. [10%]

(ii) Find the stress measure that is work-conjugate to the stretch $\mathbf{U}$ (recalling that $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ is symmetric).
Hint: $\mathbf{P}\mathbf{F}^T$ is symmetric. [30%]

END OF PAPER

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ENGINEERING TRIPOS PART IIB

Module 4C9 Continuum Mechanics

Data sheet

Indicial notation

A repeated index implies summation

$\boldsymbol{a} = a_i \boldsymbol{e}_i \qquad \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ can be written as $c_i = e_{ijk} a_j b_k$

$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ can be written as $A_{ij} = a_i b_j$

Kronecker delta: $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$

Note that $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

Permutation symbol: $e_{ijk} = 1$ when i, j, k are in cyclic order

$e_{ijk} = -1$ when i, j, k are in anti-cyclic order

$e_{ijk} = 0$ when any indices repeat

$e - \delta$ identity: $e_{ijk} e_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$

$\text{grad}\,\phi = \nabla \phi = \phi_{,i} \boldsymbol{e}_i$

$\text{div}\,\boldsymbol{v} = \nabla \cdot \boldsymbol{v} = v_{i,i}$

$\text{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v} = e_{ijk} v_{k,j} \boldsymbol{e}_i$

Gauss's theorem (the divergence theorem):

$$\int_V \frac{\partial A_{ij}}{\partial x_j} dV = \oint_S A_{ij} n_j \, dS$$

Stokes's theorem:

$$\int_S e_{ijk} \frac{\partial A_{pk}}{\partial x_j} n_i dS = \oint_C A_{pk} \; dx_k$$

Isotropic linear elasticity

Equilibrium: $\sigma_{ij,j} + b_i = 0$, $\sigma_{ij} = \sigma_{ji}$

Compatibility: $\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$

Constitutive relationships: $\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

Lame's constants: $\mu = G = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

The strain energy density U is given by: $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$

At equilibrium, the potential energy Π is minimised. Hence, for any small kinematically admissible perturbation δu_i :

$$\delta\Pi = \int_V \delta U dV - \int_S t_i^e \delta u_i dS - \int_V b_i \delta u_i dV = 0$$

Definitions: σ_{ij} is the stress tensor, ε_{ij} is the infinitesimal strain tensor, b_i is the body force vector, t_i^e is the external traction vector and u_i is the displacement vector.

Isotropic linear viscoelasticity

Relaxation modulus, $E_r(t)$:

if $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\sigma(t) = \varepsilon_0 E_r(t)$

Creep compliance, $J_c(t)$:

if $\sigma(t) = \sigma_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\varepsilon(t) = \sigma_0 J_c(t)$

The Laplace transforms of $E_r(t)$ and $J_c(t)$ are related by: $\bar{E}_r(s)\,\bar{J}_c(s) = \frac{1}{s^2}$

Boltzmann superposition principle in 1D:

$$\sigma(t) = \int_0^t \frac{\partial \varepsilon(\tau)}{\partial \tau} E_r(t-\tau) d\tau$$

$$\varepsilon(t) = \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J_c(t-\tau) d\tau$$

Correspondence principle: in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \to s\bar{E}_r(s)$, $\nu \to s\bar{\nu}_r(s)$ (for any time-dependent moduli).