4C9 Continuum Mechanics Crib: 2025

1. (a) i. The beam is slender, so Euler-Bernoulli kinematics is a good model: rotation of planar cross sections. If the transverse deflection of the centre line is $w(x_1)$, the anti-clockwise rotation of the cross-section $\phi(x_1) = dw/dx_1$, and the displacement of a material point:

$$\mathbf{u}\left(x_1, x_2\right) = w(x_1)\mathbf{e}_2 - \phi(x_1)x_2\mathbf{e}_1 = w\left(x_1\right)\mathbf{e}_2 - \frac{dw\left(x_1\right)}{dx_1}x_2\mathbf{e}_1$$

Superimpose a uniform (across the beam cross section at any point x_1) axial deformation due to the components of the external loading in this direction:

$$\mathbf{u}\left(x_1, x_2\right) = h\left(x_1\right)\mathbf{e}_1$$

The total displacement is:

$$\mathbf{u}\left(x_1, x_2\right) = w\left(x_1\right)\mathbf{e}_2 + \left[h\left(x_1\right) - \frac{dw\left(x_1\right)}{dx_1}x_2\right]\mathbf{e}_1$$

ii. The kinematics model gives zero shear strain. The beam is slender, so $\sigma_{22}\varepsilon_{22} \approx 0$ and $\sigma_{33}\varepsilon_{33} \approx 0$. Hence:

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{11}\varepsilon_{11} = \frac{1}{2}E(x_1)\varepsilon_{11}^2$$

iii. Method of minimum PE, noting that the properties are spatially varying, $E(x_1)$ and $\rho(x_1)$. At equilibrium:

$$\delta\Pi = \int_V \delta U \; dV - \int_S t_i^e \delta u_i \; dS - \int_V b_i \delta u_i \; dV = 0$$

<u>First term:</u> Using displacements

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = -w''(x_1)x_2 + h'(x_1)$$

$$\therefore U = \frac{1}{2}E(x_1)\left[w''(x_1)^2x_2^2 + h'(x_1)^2 - 2h'(x_1)w''(x_1)x2\right]$$

Variation in U with arbitrary perturbation of w and h:

$$\begin{aligned}\delta U &= \frac{\partial U}{\partial w''}\delta w'' + \frac{\partial U}{\partial h'}\delta h' \\ &= E(x_1)\left[w''(x_1)x_2^2\delta w'' + h'(x_1)\delta h' - w''(x_1)x_2\delta h' - h'(x_1)x_2\delta w''\right]\end{aligned}$$

Integrate over the beam volume, noting deformation is uniform across the width B at any point:

$$\int_V \delta U \; dV = \int_0^L \int_{-D/2}^{D/2} E(x_1)[w''(x_1)x_2^2\delta w'' + h'(x_1)\delta h' - w''(x_1)x_2\delta h' - h'(x_1)x_2\delta w'']B \; dx_2 \; dx_1$$

Integrate with respect to x_2 first. Note that:

$$\int_{-D/2}^{D/2} (\cdots)\, dx_2 = (\cdots)D \qquad \int_{-D/2}^{D/2} (\cdots)x_2\, dx_2 = 0 \qquad \int_{-D/2}^{D/2} (\cdots)x_2^2\, dx_2 = (\cdots)\frac{D^3}{12}$$

And let $I = BD^3/12$ and $A = BD$:

$$\int_V \delta U\, dV = \int_0^L E(x_1)\left[Iw''(x_1)\delta w'' + Ah'(x_1)\delta h'\right]\, dx_1$$

Integrate by parts (not forgetting that E is a function of x_1):

$$\int_V \delta U\, dV = \left[EIw''\delta w'\right]_0^L - \int_0^L I\frac{d}{dx_1}\left(Ew''\right)\delta w'\, dx_1 + \left[EAh'\delta h\right]_0^L - \int_0^L A\frac{d}{dx_1}\left(Eh'\right)\delta h\, dx_1$$

Integrate by parts again:

$$\int_V \delta U\, dV = \left[EIw''\delta w'\right]_0^L - \left[I\frac{d}{dx_1}\left(Ew''\right)\delta w\right]_0^L + \int_0^L I\frac{d^2}{dx_1^2}\left(Ew''\right)\delta w\, dx_1 + \left[EAh'\delta h\right]_0^L - \int_0^L A\frac{d}{dx_1}\left(Eh'\right)\delta h\, dx_1$$

Second term: at the tip of the beam

$$\int_S t_i^e dS = -P\cos\theta\delta w(L) - P\sin\theta\delta h(L)$$

Third term: self weight

$$\int_V b_i\delta u_i\, dV = \int_0^L \int_{-D/2}^{D/2} \left[-\rho(x_1)g\cos\theta\delta w - \rho(x_1)g\sin\theta(\delta h - \delta w' x_2)\right] B\, dx_2\, dx_1$$

Integrate with respect to x_2 first (see above).

$$\int_V b_i\delta u_i\, dV = \int_0^L \left[-\rho(x_1)gA\cos\theta\delta w - \rho(x_1)gA\sin\theta\delta h\right]\, dx_1$$

Gathering terms:

$$\delta\Pi = \int_0^L \left[I\frac{d^2}{dx_1^2}\left(Ew''\right) + \rho(x_1)gA\cos\theta\right]\delta w\, dx_1 + \int_0^L \left[-A\frac{d}{dx_1}\left(Eh'\right) + \rho(x_1)gA\sin\theta\right]\delta h\, dx_1 + \left[EIw''\delta w'\right]_0^L - \left[I\frac{d}{dx_1}\left(Ew''\right)\delta w\right]_0^L + \left[EAh'\delta h\right]_0^L + P\cos\theta\delta w(L) + P\sin\theta\delta h(L) = 0$$

So, for arbitrary δw and δh, across the length of the beam:

$$\frac{d^2}{dx_1^2}\left(EIw''\right) = -\rho(x_1)gA\cos\theta \tag{1}$$

$$\frac{d}{dx_1}\left(Eh'\right) = \rho(x_1)g\sin\theta \tag{2}$$

Boundary conditions at the tip ($x_1 = L$), for arbitrary δw, $\delta w'$ and δh:

$$w''(L) = 0 \tag{3}$$

$$\frac{d}{dx_1}\left(EIw''\right) = P\cos\theta \tag{4}$$

$$AE(L)h'(L) = -P\sin\theta \tag{5}$$

Boundary conditions at the root ($x_1 = 0$), by inspection: w = w' = h = 0.

(b) i. Correspondence principle: you can map the elastic solution to the viscoelastic solution by taking Laplace transforms and making the substitution $E \rightarrow s\bar{E}_r(s)$, as long as the boundary conditions are not time-dependent.

ii. Elastic solution for $h(x_1)$, with spatially uniform properties. Integrate (2):

$$h'' = \frac{1}{E}\sin\theta\rho g \quad \rightarrow \quad h' = \frac{1}{E}\sin\theta\rho g x_1 + C_1$$

Boundary condition (5):

$$C_1 = -\frac{1}{AE}P\sin\theta - \frac{1}{E}\sin\theta\rho g L \quad \rightarrow \quad h' = \frac{1}{E}\sin\theta\rho g(x_1 - L) - \frac{1}{AE}P\sin\theta$$

Integrate again:

$$h = \frac{1}{E}\sin\theta\rho g(\frac{1}{2}x_1^2 - Lx_1) - \frac{1}{AE}P\sin\theta x_1 + C_2$$

Boundary condition $h = 0$ at $x_1 = 0 : C_2 = 0$. Tip deflection is therefore:

$$h(L) = -\frac{1}{E}\sin\theta\left(\frac{\rho g L^2}{2} + \frac{PL}{A}\right)$$

Take Laplace transforms:

$$\bar{h}(s) = -\frac{1}{sE}\sin\theta\left(\frac{\rho g L^2}{2} + \frac{PL}{A}\right)$$

To get the viscoelastic solution substitute $E \rightarrow s\bar{E}_r(s)$:

$$\bar{h}(s) = -\frac{1}{s^2\bar{E}_r(s)}\sin\theta\left(\frac{\rho g L^2}{2} + \frac{PL}{A}\right)$$

Take the Laplace transform of the relaxation modulus:

$$E_r(t) = E_0 e^{\frac{-E_0 t}{\eta}} \quad \rightarrow \quad \bar{E}_r(s) = \frac{E_0}{s + \frac{E_0}{\eta}}$$

Substitute in:

$$\bar{h}(s) = -\left(\frac{1}{E_0 s} + \frac{1}{\eta s^2}\right)\sin\theta\left(\frac{\rho g L^2}{2} + \frac{PL}{A}\right)$$

Inverse Laplace transform:

$$h(L,t) = -\left(\frac{1}{E_0} + \frac{t}{\eta}\right)\sin\theta\left(\frac{\rho g L^2}{2} + \frac{PL}{A}\right)$$

2. (a) i. Strain components:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = -ax_3 \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -cx_3 \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\nu}{1-\nu}(a+c)x_3$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -\frac{1}{2}bx_3 - \frac{1}{2}bx_3 = -bx_3$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = -\frac{1}{2}(ax_1 + bx_2) + \frac{1}{2}(ax_1 + bx_2) = 0$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = -\frac{1}{2}(bx_1 + cx_2) + \frac{1}{2}(bx_1 + cx_2) = 0$$

The compatibility equation in 3D is (from the data sheet):

$$\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$$

As all strain components are either zero or linear in x_3, these derivatives must be zero for any choice of indices. So, the strain field is compatible for any values of a, b and c.

ii. Linear elastic constitutive equations in 3D (data sheet):

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-\nu)}\varepsilon_{kk}\delta_{ij}$$

The stress components are therefore:

$$\sigma_{11} = \frac{E}{1+\nu}(-ax_3) + \frac{\nu E}{(1+\nu)(1-\nu)}\left(-ax_3 - cx_3 + \frac{\nu}{1-\nu}(a+c)x_3\right)$$

$$= \frac{E}{1+\nu}(-ax_3) - \frac{\nu E}{(1+\nu)(1-\nu)}(a+c)x_3$$

$$\sigma_{22} = \frac{E}{1+\nu}(-cx_3) - \frac{\nu E}{(1+\nu)(1-\nu)}(a+c)x_3$$

$$\sigma_{33} = \frac{\nu E}{(1+\nu)(1-\nu)}(a+c)x_3 - \frac{\nu E}{(1+\nu)(1-\nu)}(a+c)x_3 = 0$$

$$\sigma_{12} = \frac{E}{1+\nu}(-bx_3) \qquad \sigma_{13} = \sigma_{23} = 0$$

For equilibrium in the absence of body forces:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

All non-zero stress components are linear in x_3, so this is zero for any combination of indices $i, j = 1, 2$. Also, stress components $\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = \sigma_{33} = 0$. So equilibrium is satisfied for any values of the indices, for any values of a, b and c.

(Note that this displacement field is the general solution for a thin plate in the x_1-x_2 plane, plane stress, in bending. The constants a, b and c would be found from the boundary conditions.)

(b) i. $F = \partial x_i / \partial X_J$, $\nabla_0 Q = \dfrac{\partial Q}{\partial X_i} = \dfrac{\partial x_j}{\partial X_i}\dfrac{\partial q}{\partial x_j} = F^T \nabla q$

ii. On the spatial configuration:

$$\int_{\Omega} \nabla \cdot v \,\mathrm{d}\Omega = \int_{\partial\Omega} v \cdot n \,\mathrm{d}\Gamma$$

Note that if $v = J^{-1}FV$, then $V = JF^{-1}v$. On the reference

$$\begin{aligned}\int_{\Omega_0} \nabla_0 \cdot V \,\mathrm{d}\Omega_0 &= \int_{\partial\Omega_0} V \cdot N \,\mathrm{d}\Gamma_0 \\ &= \int_{\partial\Omega_0} (JF^{-1}v) \cdot N \,\mathrm{d}\Gamma_0 \\ &= \int_{\partial\Omega_0} v \cdot (JF^{-T}N) \,\mathrm{d}\Gamma_0\end{aligned}$$

Since the divergence is preserved,

$$\int_{\partial\Omega} v \cdot n \,\mathrm{d}\Gamma = \int_{\partial\Omega_0} V \cdot N \,\mathrm{d}\Gamma_0 = \int_{\partial\Omega_0} v \cdot (JF^{-T}N) \,\mathrm{d}\Gamma_0,$$

hence

$$n \,\mathrm{d}\Gamma = JF^{-T}N \,\mathrm{d}\Gamma_0,$$

which is Nanson's formula.

iii. Note that if $v = J^{-1}FV$, then $V = JF^{-1}v$. Using integration by parts and change-of-variables:

$$\begin{aligned}\int_{\Omega} (\nabla \cdot v) q \,\mathrm{d}\Omega &= -\int_{\Omega} v \cdot \nabla q \,\mathrm{d}\Omega \quad \text{(by integration by parts, } q = 0 \text{ on } \partial\Omega) \\ &= -\int_{\Omega} v \cdot (F^{-T}\nabla_0 Q) \,\mathrm{d}\Omega \\ &= -\int_{\Omega_0} v \cdot (F^{-T}\nabla_0 Q) J \,\mathrm{d}\Omega_0 \\ &= -\int_{\Omega_0} (JF^{-1}v) \cdot (\nabla_0 Q) \,\mathrm{d}\Omega_0 \\ &= \int_{\Omega_0} (\nabla_0 \cdot (JF^{-1}v))\, Q \,\mathrm{d}\Omega_0 \quad \text{(by integration by parts, } Q = 0 \text{ on } \partial\Omega) \\ &= \int_{\Omega_0} (\nabla_0 \cdot V)\, Q \,\mathrm{d}\Omega_0.\end{aligned}$$

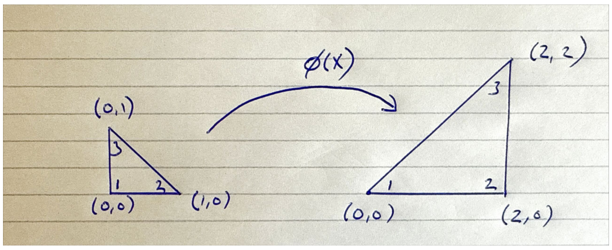
Since $q(x) = Q(X)$ is zero on the boundary but is otherwise arbitrary,

$$\int_{\Omega} \nabla \cdot v \,\mathrm{d}\Omega = \int_{\Omega_0} \nabla_0 \cdot V \,\mathrm{d}\Omega_0,$$

which proves that the divergence is preserved.

3. (a) i. Deformation gradient is constant, therefore transformation is linear (there are more complex maps that would still give the same mapping of the vertices, but $\det F$ would not be constant).

$$\phi(X) = \begin{bmatrix} 2X_1 + 2X_2 \\ 2X_2 \end{bmatrix}$$



ii. Deformation gradient is constant, therefore determinant is constant. Reference area is 1/2, after mapping the area is 2. Therefore $\det F = 4$ since $\mathrm{d}\Omega = J\,\mathrm{d}\Omega_0$.

iii. $F = \nabla_0 \phi = \dfrac{\partial \phi_i}{\partial X_j}$. Hence

$$F = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

Verify determinant: $\det F = 4$.

iv. Vectors (give unit length): $W_t = [-1/\sqrt{2}, 1/\sqrt{2}]^T$, $W_n = [1/\sqrt{2}, 1/\sqrt{2}]^T$
Compute F^{-1}:

$$F^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

and noting that $(\det F)^{-1} = 1/4$.
Applying the first map to W_t

$$w_t^{(1)} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/(2\sqrt{2}) \end{bmatrix}$$

Applying the first map to W_n

$$w_n^{(1)} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/(2\sqrt{2}) \end{bmatrix}$$

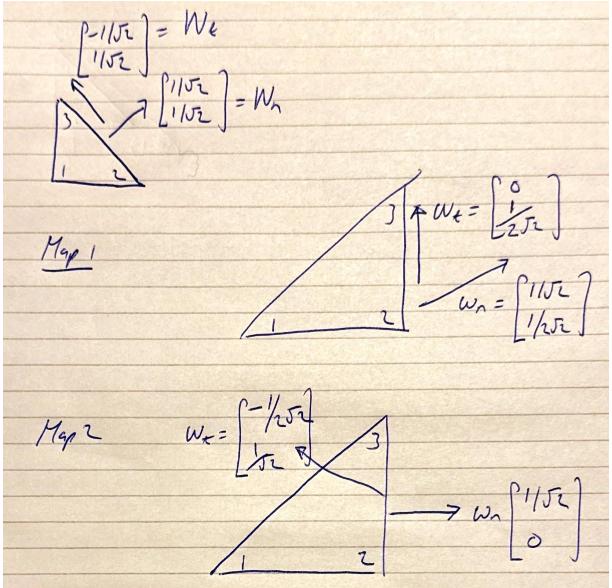
Applying the second map to W_t

$$w_t^{(2)} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/(2\sqrt{2}) \\ 1/\sqrt{2} \end{bmatrix}$$

Applying the second map to W_n

$$w_n^{(2)} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}$$

v. Plot:



Noteworthy is that:

- Map 1 maintains the tangent direction of the tangential field but the magnitude is different.
  A more subtle point is that for the normal field, while the mapped field is no longer normal, the component normal to the edge times the edge length is preserved.
- Map 2 maintains the normal direction of the normal field but the magnitude is different.
  A more subtle point is that for the tangential field, while the mapped field is not tangential, the component tangential to the edge times the edge length is preserved.

(b) A hyperelastic constitutive model is defined by a strain energy density function that is time-independent and includes no dissipative terms. Stress are computed by taking derivatives with respect to a deformation measure.

The strain energy density function can only depend on quantities that are invariant with respect to the reference frame and they must eliminate rigid body rotations.

(c)

$$\frac{Dq}{Dt} = \left.\frac{\partial q(\phi(X,t),t)}{\partial t}\right|_{X=\text{const}} = \frac{\partial q}{\partial t} + \nabla q \cdot \frac{\partial \phi}{\partial t} = \frac{\partial q}{\partial t} + \nabla q \cdot v$$

From $\rho_0 = J\rho$,

$$\frac{D\rho_0}{Dt} = \frac{D(J\rho)}{Dt} = \rho\frac{DJ}{Dt} + \frac{D\rho}{Dt}J = 0$$

Inserting $\dot{J} = J\nabla \cdot v$,

$$J\rho\nabla \cdot v + J\frac{D\rho}{Dt} = 0$$

leading to

$$\rho\nabla \cdot v + \frac{D\rho}{Dt} = 0$$

Using the material derivative for a scalar,

$$\frac{\partial \rho}{\partial t} + \nabla\rho \cdot v + \rho\nabla \cdot v = 0$$

Grouping terms two and three,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

Ver.2