EGT3
ENGINEERING TRIPOS PART IIB

Friday 9 May 2025   2 to 3.40

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C9 Continuum Mechanics datasheet (2 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A slender cantilever beam is inclined at an angle θ to the horizontal, as shown in Fig. 1. The beam has length L, depth D and out-of-plane width B. The position of a material point on the mid-plane of the beam is $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. The loading is provided by self-weight and a vertical tip force of magnitude P, in the directions shown in the figure. Infinitesimal deformations can be assumed.

(a) The beam is manufactured from a graded linear elastic material with density $\rho(x_1)$ and Young's modulus $E(x_1)$ that vary with distance x_1 along the beam.

(i) Show that the displacement field for the mid-plane can be approximated by

$$\mathbf{u}(x_1, x_2) = w\mathbf{e}_2 + \left(h - \frac{dw}{dx_1}x_2\right)\mathbf{e}_1,$$

where $h(x_1)$ and $w(x_1)$ are the displacements of points on the centre-line of the beam in directions $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. [10%]

(ii) Explain why the elastic strain energy density at a point depends only on the stress component σ_{11} and the strain component ε_{11}. [10%]

(iii) Hence, use the method of minimum potential energy to derive governing equations for the centre-line deflections $h(x_1)$ and $w(x_1)$, and for the boundary conditions. A solution for the deflected shape is not required. [40%]

(b) The beam is instead manufactured from a spatially uniform linear viscoelastic material with density ρ and relaxation modulus $E_r(t) = E_0 \exp(-E_0 t/\eta)$.

(i) Explain briefly what is meant by the *correspondence principle*. [10%]

(ii) Using the correspondence principle, derive an expression for the time-dependent axial deflection at the tip of the cantilever, $h(L, t)$. [30%]

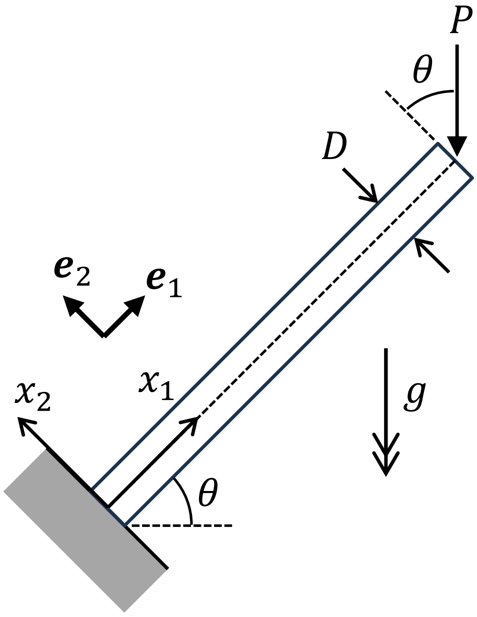

Fig. 1

2 (a) A linear elastic body with Young's modulus E and Poisson's ratio ν undergoes infinitesimal deformation. The displacement field $\mathbf{u}(\mathbf{x})$, where the position $\mathbf{x} = x_i \mathbf{e}_i$, has components

$$
\begin{aligned}
u_1 &= -\left(a x_1 + b x_2\right) x_3, \\
u_2 &= -\left(b x_1 + c x_2\right) x_3, \\
u_3 &= \frac{1}{2}\left[a x_1^2 + 2 b x_1 x_2 + c x_2^2 + \frac{\nu}{1-\nu}(a+c) x_3^2\right],
\end{aligned}
$$

where a, b and c are constants. Prove any requirements on a, b and c to ensure that:

(i) the deformation is compatible; and [25%]

(ii) the body is in equilibrium in the absence of body forces. [25%]

(b) Consider a body with reference configuration Ω_0 and a current configuration Ω. The deformation gradient is $\boldsymbol{F}$ and its determinant $J = \det \boldsymbol{F}$.

(i) Show that $\nabla_0 Q = \boldsymbol{F}^T \nabla q$, where Q is a scalar field on a reference configuration and q is its push-forward to the spatial configuration. [10%]

(ii) Consider a vector $\boldsymbol{V}$ on the reference configuration. If the push-forward to the spatial configuration is given by $\boldsymbol{v} = J^{-1} \boldsymbol{F} \boldsymbol{V}$, divergences of the vector fields are preserved, i.e. $\int_{\Omega_0} \nabla_0 \cdot \boldsymbol{V} \, d\Omega_0 = \int_{\Omega} \nabla \cdot \boldsymbol{v} \, d\Omega$. Use this result to find the relationship between the outward unit normal vectors to Ω_0 and to Ω. [20%]

(iii) Prove that $\int_{\Omega_0} \nabla_0 \cdot \boldsymbol{V} \, d\Omega_0 = \int_{\Omega} \nabla \cdot \boldsymbol{v} \, d\Omega$ when $\boldsymbol{v} = J^{-1} \boldsymbol{F} \boldsymbol{V}$.

Hint: start with $\int_{\Omega} (\nabla \cdot \boldsymbol{v}) q \, d\Omega$, where q is a differentiable function that goes to zero on the boundary of Ω. [20%]

3 (a) Consider a triangle in the reference configuration with vertices $\hat{v}_1 = (0,0)$, $\hat{v}_2 = (1,0)$, $\hat{v}_3 = (0,1)$. Under a transformation the vertices map to $v_1 = (0,0)$, $v_2 = (2,0)$, $v_3 = (2,2)$. The deformation gradient, $\boldsymbol{F}$, is constant.

(i) Sketch the two configurations and give the deformation map $\boldsymbol{\phi}(\boldsymbol{X})$. [20%]

(ii) By geometric arguments alone, give $\det \boldsymbol{F}$. [10%]

(iii) Compute the deformation gradient. [10%]

(iv) Consider two vector fields, $\boldsymbol{W}_t$ and $\boldsymbol{W}_n$, on the reference configuration. On the edge $\hat{v}_2$–$\hat{v}_3$, $\boldsymbol{W}_t$ is tangential to the edge and $\boldsymbol{W}_n$ is normal to the edge. Apply the transformations $\boldsymbol{w}^{(1)} = (\det \boldsymbol{F})^{-1} \boldsymbol{F} \boldsymbol{W}$ and $\boldsymbol{w}^{(2)} = \boldsymbol{F}^{-T} \boldsymbol{W}$, which are both push-forward transformations, to $\boldsymbol{W}_t$ and $\boldsymbol{W}_n$ at a point on the $\hat{v}_2$–$\hat{v}_3$ edge. [20%]

(v) Sketch the vectors $\boldsymbol{W}_t$ and $\boldsymbol{W}_n$ at a point on the $\hat{v}_2$–$\hat{v}_3$ edge, and on the spatial configuration for the $\boldsymbol{w}^{(1)}$ and $\boldsymbol{w}^{(2)}$ push-forward transformations. Comment on any significant features. [10%]

(b) What properties make a constitutive model hyperelastic? Comment on any requirements on the kinematic quantities that a constitutive model may depend on. [10%]

(c) Derive the expression for the material time derivative of a scalar quantity on the current configuration, and use this to prove that conservation of mass requires that $\partial \rho / \partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0$, where ρ is the density and $\boldsymbol{v}$ is the velocity.

Note: $\dot{J} = J \nabla \cdot \boldsymbol{v}$. [20%]

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ENGINEERING TRIPOS PART IIB

Module 4C9 Continuum Mechanics

Data sheet

Indicial notation

A repeated index implies summation

$\boldsymbol{a} = a_i \boldsymbol{e}_i \qquad \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ can be written as $c_i = e_{ijk} a_j b_k$

$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ can be written as $A_{ij} = a_i b_j$

Kronecker delta: $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$

Note that $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

Permutation symbol: $e_{ijk} = 1$ when i, j, k are in cyclic order

$e_{ijk} = -1$ when i, j, k are in anti-cyclic order

$e_{ijk} = 0$ when any indices repeat

$e - \delta$ identity: $e_{ijk} e_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$

$\text{grad}\, \phi = \nabla \phi = \phi_{,i} \boldsymbol{e}_i$

$\text{div}\, \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = v_{i,i}$

$\text{curl}\, \boldsymbol{v} = \nabla \times \boldsymbol{v} = e_{ijk} v_{k,j} \boldsymbol{e}_i$

Gauss's theorem (the divergence theorem):

$$\int_V \frac{\partial A_{ij}}{\partial x_j} dV = \oint_S A_{ij} n_j \, dS$$

Stokes's theorem:

$$\int_S e_{ijk} \frac{\partial A_{pk}}{\partial x_j} n_i dS = \oint_C A_{pk} \, dx_k$$

Isotropic linear elasticity

Equilibrium: $\sigma_{ij,j} + b_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji}$

Compatibility: $\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$

Constitutive relationships: $\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

Lame's constants: $\mu = G = \frac{E}{2(1+\nu)} \quad , \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

The strain energy density U is given by: $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$

At equilibrium, the potential energy Π is minimised. Hence, for any small kinematically admissible perturbation δu_i :

$$\delta\Pi = \int_V \delta U dV - \int_S t_i^e \delta u_i dS - \int_V b_i \delta u_i dV = 0$$

Definitions: σ_{ij} is the stress tensor, ε_{ij} is the infinitesimal strain tensor, b_i is the body force vector, t_i^e is the external traction vector and u_i is the displacement vector.

Isotropic linear viscoelasticity

Relaxation modulus, $E_r(t)$:

if $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\sigma(t) = \varepsilon_0 E_r(t)$

Creep compliance, $J_c(t)$:

if $\sigma(t) = \sigma_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\varepsilon(t) = \sigma_0 J_c(t)$

The Laplace transforms of $E_r(t)$ and $J_c(t)$ are related by: $\bar{E}_r(s)\,\bar{J}_c(s) = \frac{1}{s^2}$

Boltzmann superposition principle in 1D:

$$\sigma(t) = \int_0^t \frac{\partial \varepsilon(\tau)}{\partial \tau} E_r(t-\tau) d\tau$$

$$\varepsilon(t) = \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J_c(t-\tau) d\tau$$

Correspondence principle: in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \rightarrow s\bar{E}_r(s)$, $\nu \rightarrow s\bar{\nu}_r(s)$ (for any time-dependent moduli).