4C9 Continuum Mechanics Crib

2024

1. (a) i. The tank is open-ended, so there will be no longitudinal stress, σ_{22}. Because it is thin-walled, it can be assumed to be in a state of plane stress, so there will be no through-thickness stress, σ_{33}, or out-of plane shear σ_{23} or σ_{13}. The only loading is internal pressure, so there will be no in-plane shear, σ_{12}. The pressure is balanced by the hoop stress σ_{11} only.

The elastic strain energy per unit volume at any point in the wall

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{11}\varepsilon_{11} = \frac{1}{2}E\varepsilon_{11}^2$$

using the constitutive model. The total elastic strain in the wall is therefore

$$\int_V U\,dV = \int_0^H \frac{1}{2}E\varepsilon_{11}^2\,2\pi Rw\,dz$$

noting that $\varepsilon_{11}(z)$ varies with position.

ii. We are using the method of minimum potential energy to derive an expression for the hoop strain, so we can use $\varepsilon_{11}(z)$ as the kinematic variable; there is no need to work in terms of displacements. The variation in potential energy:

$$\begin{aligned}\delta\Pi &= \int_V \delta U\,dV - \int_S t_i^e \delta u_i\,dS \\ &= \int_0^H \frac{\partial}{\partial\varepsilon_{11}}\left(\frac{1}{2}E\varepsilon_{11}^2\right)\delta\varepsilon_{11}\,2\pi Rw\,dz - \int_0^H \rho g(H-z)\delta R\,2\pi R\,dz\end{aligned}$$

Noting that $\delta R = R\delta\varepsilon_{11}$ this reduces to

$$\delta\Pi = \int_0^H \left[E\varepsilon_{11}w - \rho g(H-z)R\right]\delta\varepsilon_{11}2\pi R\,dz$$

For $\delta\Pi = 0$ for any $\delta\varepsilon_{11}$, then

$$\varepsilon_{11} = \frac{\rho g(H-z)R}{Ew}$$

Note that, applying the constitutive model, this recovers the familiar equilibrium relationship for the hoop stress in a thin walled cylinder: $\sigma_{11} = \rho g(H-z)R/w$

(b) i. The uniaxial response of the material is given by the relationship

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Relaxation modulus $E_r(t)$: consider a step in strain of magnitude ε_0.

$t = 0 \quad \dot{\varepsilon} >> \varepsilon \quad \dot{\varepsilon}0 = \frac{\dot{\sigma}}{E} \quad \therefore \varepsilon = \frac{\sigma}{E}\,, \quad \sigma(0) = E\varepsilon_0$

$t > 0 \quad \dot{\varepsilon} = 0 \quad \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = 0 \quad \therefore \sigma = (E\varepsilon_0)\exp\left(-\frac{Et}{\eta}\right)$

$E_r(t) = \frac{\sigma(t)}{\varepsilon_0} = E\exp\left(-\frac{Et}{\eta}\right)$

Creep compliance $J_c(t)$: consider a step in stress of magnitude σ_0.

$t = 0 \quad \dot{\sigma} >> \sigma \quad \dot{\varepsilon} = \frac{\dot{\sigma}}{E} \quad \therefore \varepsilon = \frac{\sigma}{E}\,, \quad \varepsilon(0) = \frac{\sigma_0}{E}$

$t > 0 \quad \dot{\sigma} = 0 \quad \dot{\varepsilon} = \frac{\sigma_0}{\eta} \quad \therefore \varepsilon = \frac{\sigma_0 t}{\eta} + \frac{\sigma_0}{E}$

$J_c(t) = \frac{\varepsilon(t)}{\sigma_0} = \frac{t}{\eta} + \frac{1}{E}$

ii. The tank is filled with water at a constant rate $\dot{H}$, such that $H = \dot{H}t$. The constitutive model gives the hoop strain rate (all other stress components are zero):

$$\varepsilon_{11}(t) = \int_0^t J_c(t-\tau)\frac{\partial\sigma_{11}(\tau)}{\partial\tau}\,d\tau$$

At a given height z the hoop stress, deduced from part (a) or otherwise, will be

$$t < z/\dot{H}: \quad \sigma_{11} = 0$$

$$t \geq z/\dot{H}: \quad \sigma_{11} = \frac{\rho g(\dot{H}t - z)R}{w}$$

Substituting for the creep compliance and the hoop stress into the constitutive equation:

$$t < z/\dot{H}: \quad \varepsilon_{11}(t) = 0$$

$$\begin{aligned} t \geq z/\dot{H}: \quad \varepsilon_{11}(t) &= \int_{z/\dot{H}}^{t} \left(\frac{t-\tau}{\eta} + \frac{1}{E}\right)\frac{\partial}{\partial\tau}\left(\frac{\rho g(\dot{H}\tau - z)R}{w}\right) d\tau \\ &= \int_{z/\dot{H}}^{t} \left(\frac{t-\tau}{\eta} + \frac{1}{E}\right)\frac{\rho g\dot{H}R}{w}\,d\tau \\ &= \left[\left(\frac{t\tau}{\eta} - \frac{\tau^2}{2\eta} + \frac{\tau}{E}\right)\frac{\rho g\dot{H}R}{w}\right]_{z/\dot{H}}^{t} \\ &= \left[\left(\frac{t^2}{2\eta} + \frac{t}{E}\right) - \left(\frac{tz}{\dot{H}\eta} - \frac{z^2}{2\eta\dot{H}^2} + \frac{z}{E\dot{H}}\right)\right]\frac{\rho g\dot{H}R}{w} \end{aligned}$$

Rate of change of the radius ($t \geq z/\dot{H}$):

$$\dot{R} = \dot{\varepsilon}_{11} R = \left[\frac{t}{\eta} + \frac{1}{E} - \frac{z}{\dot{H}\eta} \right] \frac{\rho g \dot{H} R^2}{w}$$

From the constitutive model $\varepsilon_{33} = -\nu \varepsilon_{11}$, where ν is time independent, so

$$\dot{w} = \dot{\varepsilon}_{33} w = -\nu \dot{\varepsilon}_{11} w = -\nu \left[\frac{t}{\eta} + \frac{1}{E} - \frac{z}{\dot{H}\eta} \right] \rho g \dot{H} R$$

2. (a) The principle of virtual work

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\; dV - \int_S t_i^e \delta u_i\; dS - \int_V b_i \delta u_i\; dV = 0$$

Using $\delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, and the symmetry of the stress tensor:

$$\int_V \sigma_{ij}\delta u_{i,j}\; dV - \int_S t_i^e \delta u_i\; dS - \int_V b_i \delta u_i\; dV = 0$$

Using $\frac{\partial}{\partial x_j}(\sigma_{ij}\delta u_i) = \sigma_{ij}\delta u_{i,j} + \sigma_{ij,j}\delta u_i$ gives:

$$\int_V \frac{\partial}{\partial x_j}(\sigma_{ij}\delta u_i)\; dV - \int_V \sigma_{ij,j}\delta u_i\; dV - \int_S t_i^e \delta u_i\; dS - \int_V b_i \delta u_i\; dV = 0$$

Applying the divergence theorem:

$$\int_S (\sigma_{ij}\delta u_i) n_j\; dS - \int_V \sigma_{ij,j}\delta u_i\; dV - \int_S t_i^e \delta u_i\; dS - \int_V b_i \delta u_i\; dV = 0$$

For this to hold for any δu_i requires $\sigma_{ij,j} + b_i = 0$ within V and $t_i^e = \sigma_{ij} n_j$ across S.

(b) The balance of moments on the body:

$$\int_S \mathbf{x} \times \mathbf{t^e}\; dS + \int_V \mathbf{x} \times \mathbf{b}\; dV = 0$$

or in indicial notation (using $t_i^e = \sigma_{ij} n_j$):

$$\int_S e_{ijk}\; x_j\; \sigma_{kp} n_p\; dS + \int_V e_{ijk}\; x_j\; b_k\; dV = 0$$

Applying the divergence theorem:

$$\int_V \frac{\partial}{\partial x_p}(e_{ijk}\; x_j\; \sigma_{kp})\; dV + \int_V e_{ijk}\; x_j\; b_k\; dV = 0$$

$$\therefore e_{ijk}\left[\frac{\partial}{\partial x_p}(x_j\; \sigma_{kp}) + x_j\; b_k\right] = 0$$

$$\therefore e_{ijk}\left[x_j\left(\frac{\partial \sigma_{kp}}{\partial x_p} + b_k\right) + \sigma_{kj}\right] = 0$$

Using the equilibrium equation $\sigma_{ij,j} + b_i = 0$:

$$e_{ijk}\sigma_{kj} = 0$$

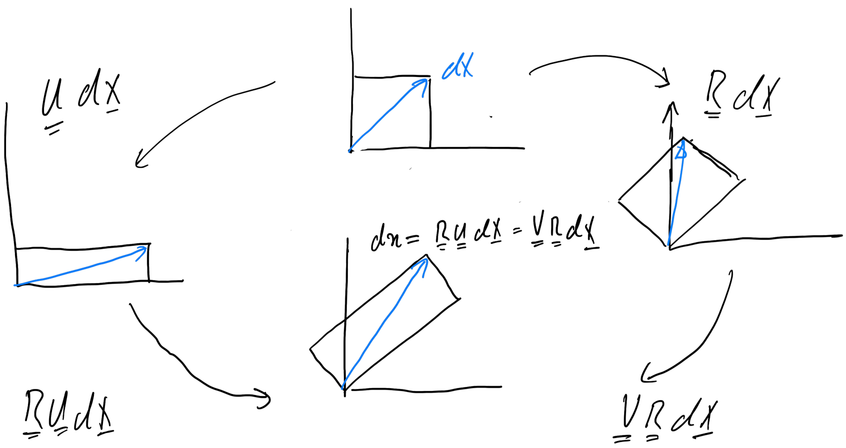
Expanding, and using the fact that $e_{ijk} = -e_{ikj}$, leads to $\sigma_{kj} = \sigma_{jk}$.

i. Consider the SVD $B = W\Sigma Z^T$, which exists for all tensors B.
With $B = W\Sigma Z^T = WZ^T Z\Sigma Z^T$, if $B = RU$ we have $R = WZ^T$ and $U = Z\Sigma Z^T$. R is by definition orthogonal. Since the entries of Σ are non-negative, U is symmetric positive-semidefinite.
For $B = VR = W\Sigma W^T WZ^T$, we have $V = W\Sigma W^T$ (SPD) and $R = WZ^T$ (as before).
If $\det B \neq 0$, Σ entries are strictly positive, hence U and V are SPD and $\det U, \det V > 0$. This leave $\det(RU) = \det R \det U > 0$, therefore $\det R = 1$ (proper orthogonal).

ii. Due to the existence of the preceding decomposition $F^TF = UR^TRU = U^2$ and $FF^T = VRR^TV = V^2$. This shows that F^TF and FF^T are suitable strain measures in that they are unaffected by rigid body rotations.

iii. For $dx = RUdX$, this gives a stretch of the incremental vector dX in the directions of the eigenvalues of U, scaled by the corresponding eigenvalues. follow by a rotation R into the spatial configuration.

For $dx = VRdX$, this rotates dX into the spatial configuration by R and the stretches the vector by V in the directions of the eigenvectors of V, scaled by the corresponding eigenvalues



3. (a) i.

$$F := \nabla_X \phi = \begin{bmatrix} p & & \\ & q & \\ & & r \end{bmatrix}$$

To be admissible, we require that $\det F > 0$, therefore it is required that $pqr > 0$.

ii.

$$F = \begin{bmatrix} 1 & \beta t^2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$\det F = 1$, therefore volume is preserved.

(b) i. Time-dependent rigid body motion.

ii. Since R is a rotation, $R^{-1} = R^T$, and

$$X = R^T(\phi - c) = R^T(x - c)$$

iii. $V(X,t) = \dot{\phi} = \dot{R}X + \dot{c}$ and $v(x,t) = V(X,t)|_{X=\phi^{-1}(x,t)}$. Inserting expression for X in terms of x,

$$v = \dot{R}R^T(x - c) + \dot{c}$$

(c) i. $F = RU$, $\dot{F} = \dot{R}U + R\dot{U}$. For a rigid body motion the stretch tensor is the identity, $U = I$ and therefore $\dot{F} = \dot{R}$. Consider $PF^TF^{-T} : \dot{F} = PR^TR : \dot{R}$. Re-arranging (use index notation), $PR^T : \dot{R}R^T$. Since $RR^T = I$,

$$\dot{\overline{RR^T}} = \dot{R}R^T + R\dot{R}^T = 0.$$

Hence $\dot{R}^TR = -R^T\dot{R}$, i.e. $\dot{R}^TR$ is skew-symmetric. By symmetry of PR^T (use hint) and skew-symmetry of $\dot{R}^TR$ the stress power is zero.

ii. $\dot{F} = \dot{R}U + R\dot{U}$, hence stress power is

$$P : \dot{F} = P : (\dot{R}U + R\dot{U}).$$

$$P : (\dot{R}U + R\dot{U}) = P : (\dot{R}U) + P : (R\dot{U})$$

Considering the second term:

$$P : (R\dot{U}) = (R^TP) : \dot{U}$$

(by index manipulations). Considering the first term and using $R^TR = I$:

$$\begin{aligned} P : (\dot{R}U) &= P : (\dot{R}R^TRU) \\ &= P : (\dot{R}R^TF) \\ &= (PF^T) : (\dot{R}R^T) \end{aligned}$$

The contraction of a symmetric tensor (PF^T here) and a skew-symmetric tensor ($\dot{R}R^T$) is zero. Therefore R^TP is the stress measure that is conjugate to U. Since U is symmetric, we could define a symmetric stress tensor $1/2(R^TP + P^TR)$.