$\boxed{Q1}$

(a) Linear elasticity : $U = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$

- Beam is slender : $\sigma_{22} = \sigma_{33} = 0$
- Kinematics : model assumes zero shear deformation, $\varepsilon_{12} = 0$ etc
- Only non-zero contributions to U is therefore σ_{11}, ε_{11}
- Linear elastic : $\sigma_{11} = E_1 \varepsilon_{11}$ $\therefore U = \frac{1}{2} \sigma_{11} \varepsilon_{11} = \underline{\frac{1}{2} E_1 \varepsilon_{11}^2}$

(b) Elastic beam only ($h_2 = 0$) :

- Kinetics (given) : $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = - w_{,11}(x_1)\, x_2$
- Strain energy density (from (a)) : $U = \frac{1}{2} E_1 (w_{,11}\, x_2)^2$
- Variation in PE (DOF = centre line deflection w) :

$$\delta \Pi = \int_V \delta U \, dV - F \delta w(x_1 = L)$$

$$= \int_V \frac{\partial U}{\partial w_{,11}} \delta w_{,11} \, dV - F \delta w(x_1 = L)$$

$$= \int_V E_1 w_{,11} x_2^2 \, \delta w_{,11} \, dV - F \delta w(x_1 = L)$$

$$= E_1 b \int_0^L \int_{-h_1}^{+h_1} w_{,11}(x_1)\, x_2^2 \, \delta w_{,11} \, dx_2 \, dx_1 - F \delta w(x_1 = L)$$

$I_1 = \frac{b(2h_1)^3}{12} \rightarrow$

$$= E_1 I_1 \int_0^L w_{,11}(x_1)\, \delta w_{,11} \, dx_1 - F \delta w(x_1 = L)$$

- Integrate by parts :

$$\delta \Pi = - E_1 I_1 \int_0^L w_{,111}(x_1)\, \delta w_{,1} \, dx_1 + E_1 I_1 \left[w_{,11}(x_1)\, \delta w_{,1} \right]_0^L - F \delta w(x_1 = L)$$

Q1 cont...

- Integrate by parts again:

$$\delta\Pi = E_1 I_1 \int_0^L w_{,1111}(x_1)\,\delta w\, dx_1 - E_1 I_1 \left[w_{,111}(x_1)\,\delta w \right]_0^L$$
$$+ E_1 I_1 \left[w_{,11}(x_1)\,\delta w_{,1} \right]_0^L - F\,\delta w(x_1 = L)$$

- Require $\delta\Pi = 0$ for any kinematically admissible δw:

$\therefore \underbrace{w_{,1111} = 0}$ for $0 \leq x_1 \leq L$

- Boundary conditions:

At $x_1 = L$: $(F + E_1 I_1 w_{,111})\,\delta w = 0 \quad \therefore \underbrace{w_{,111} = -\dfrac{F}{E_1 I_1}}$

At $x_1 = L$: $w_{,11}\,\delta w_{,1} = 0 \quad \therefore \underbrace{w_{,11} = 0}$

At $x_1 = 0$: $w_{,111}\,\delta w = 0 \rightarrow$ satisfied by B.C. $\underbrace{w(x_1 = 0) = 0}$ (given)

At $x_1 = 0$: $w_{,11}\,\delta w_{,1} = 0 \rightarrow$ satisfied by B.C. $\underbrace{w_{,1}(x_1 = 0) = 0}$ (given)

(c) Add viscoelastic layers ($h_2 \neq 0$):

Strategy: relate stress σ_{11} to strain ε_{11}, and therefore curvature κ (from kinematics); then relate stresses σ_{11} to the moment; then relate moment to the tip force

- Elastic layer:

$$\varepsilon_{11} = u_{1,1} = -w_{,1}(x_1)\,x_2 = -\kappa(x_1)\,x_2$$

$$\sigma_{11} = E_1 \varepsilon_{11} = -\kappa(x_1)\,x_2\,E_1$$

Q1 cont....

- Viscoelastic layers:

Again $\varepsilon_{11} = u_{1,1} = -K(x_1)x_2$ (same as elastic layer → given)

Use Boltzmann superposition principle to get stress:

$$\sigma_{11} = \int_0^t \frac{\partial \varepsilon_{11}(\tau)}{\partial \tau} E_r(t-\tau)\, d\tau$$

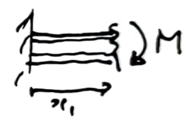
$$= -\int_0^t x_2 \frac{\partial K(x_1,\tau)}{\partial \tau} E_r(t-\tau)\, d\tau$$

Given: $E_r(t) = E_2 e^{-tE_2/\eta_2}$

Also noting: $K = \alpha t \Rightarrow K(t=0) = 0 \Rightarrow$ no initial discontinuity to allow for

- Total moment on the cross-section at position x_1:

$$M(x_1,t) = -b\int_{-(h_1+h_2)}^{+(h_1+h_2)} \sigma_{11}(x_1,x_2,t)\, x_2\, dx_2$$



- Integrate separately across elastic and visco-elastic layers:

$$M(x_1,t) = -b\int_{-h_1}^{h_1}\left[-K(x_1,t)\,x_2\,E_1\right]x_2\, dx_2$$

$$-b\int_{h_1}^{h_1+h_2}\left[-x_2\int_0^t \frac{\partial K(x_1,t)}{\partial \tau} E_r(t-\tau)\, d\tau\right]x_2\, dx_2$$

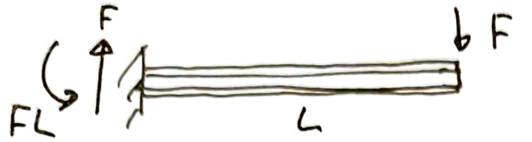
$$-b\int_{-(h_1+h_2)}^{-h_1}\left[-x_2\int_0^t \frac{\partial K(x_1,t)}{\partial \tau} E_r(t-\tau)\, d\tau\right]x_2\, dx_2$$

$$= E_1 I_1 K(x_1,t) + (I_2 - I_1)\int_0^t \frac{\partial K(x_1,t)}{\partial \tau} E_r(t-\tau)\, d\tau \quad (1)$$

where: $I_1 = \frac{b(2h_1)^3}{12}$, $I_2 = \frac{b(2h_1+2h_2)^3}{12}$

Q1 Cont...

- At the root of the cantilever $(x_1 = 0)$



$M = FL$

Given: $\kappa = \alpha t$

→ sub in to ①

solve in the time domain, or take Laplace transforms

$$\therefore F(t)L = E_1 I_1 \alpha t + (I_2 - I_1)\int_0^t \alpha E_2 e^{-(t-\tau)\frac{E_2}{\eta_2}} d\tau$$

$$= E_1 I_1 \alpha t + E_2 (I_2 - I_1)\alpha \left[\frac{\eta_2}{E_2} e^{-(t-\tau)\frac{E_2}{\eta_2}}\right]_0^t$$

$$\therefore F(t) = \frac{E_1 I_1 \alpha t}{L} + \frac{\eta_2 (I_2 - I_1)\alpha}{L}\left[1 - e^{-t\frac{E_2}{\eta_2}}\right]$$

$\boxed{Q2}$ (a)

(i) $\displaystyle\int_V \nabla \times \underline{a}\, dV = \int_V \epsilon_{ijk} \frac{\partial}{\partial x_j} a_k \, dV$

$\displaystyle = \oint_S \epsilon_{ijk} a_k n_j \, dS$ (Data Sheet $\rightarrow$ divergence theorem)

$\displaystyle = \oint_S \underline{n} \times \underline{a} \, dS$

(ii) $\displaystyle \nabla\left(\sqrt{\underline{a}\cdot\underline{a}}\right) = \frac{\partial}{\partial x_i}\left(a_j a_j\right)^{\frac{1}{2}} \underline{e}_i$ $\qquad (\underline{a} = a_i \underline{e}_i)$

$\displaystyle = \frac{1}{2}(a_j a_j)^{-\frac{1}{2}} (2a_i)\, \underline{e}_i$ $\qquad \displaystyle \frac{\partial}{\partial x_i}(\dots) = \frac{\partial}{\partial a_j}(\dots)\frac{\partial a_j}{\partial x_i} = \frac{\partial}{\partial a_j}(\dots)\,\delta_{ij}$

$\displaystyle = (a_j a_j)^{-\frac{1}{2}} a_i \underline{e}_i$

$\displaystyle = \frac{\underline{a}}{\sqrt{\underline{a}\cdot\underline{a}}}$

(iii) $\displaystyle \nabla \times (\underline{a} \times \underline{b}) = \epsilon_{ijk} \frac{\partial}{\partial x_j}\left(\epsilon_{kpq} a_p b_q\right)$

$\displaystyle = \left(\epsilon_{kij}\epsilon_{kpq}\right)\frac{\partial}{\partial x_j}\left(a_p b_q\right)$

ϵ-δ identity (Data Sheet)

$\displaystyle = \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right)\frac{\partial}{\partial x_j}\left(a_p b_q\right)$

$\displaystyle = \frac{\partial}{\partial x_j}(a_i b_j) - \frac{\partial}{\partial x_j}(a_j b_i)$

$\displaystyle = a_i \frac{\partial b_j}{\partial x_j} - b_i \frac{\partial a_j}{\partial x_j} + b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j}$

$\displaystyle = \underline{a}(\nabla\cdot\underline{b}) - \underline{b}(\nabla\cdot\underline{a}) + \underline{b}\cdot\nabla\underline{a} - \underline{a}\cdot\nabla\underline{b}$

2 b i

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial X_k}\frac{\partial X_k}{\partial x_i} = F^{-1}_{ki}\frac{\partial f}{\partial X_k} = \mathbf{F}^{-T}\nabla_0 f$$

ii

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial u_i}{\partial X_k}F^{-1}_{kj} = (\nabla_0\mathbf{u})\mathbf{F}^{-1}$$

iii

$$\begin{aligned}\int_{\Omega_0}\nabla_0\cdot(J\mathbf{F}^{-T})dV &= \int_{\partial\Omega_0}J\mathbf{F}^{-T}\mathbf{N}dS\\ &= \int_{\partial\Omega}\mathbf{n}ds\\ &= \int_{\Omega}(\nabla\cdot\mathbf{I})\mathbf{n}ds\\ &= 0\end{aligned}$$

This must hold for any domain, therefore can be localised and integrals removed, proving the result.

iv

$$\begin{aligned}\frac{\partial[J\mathbf{F}^{-1}\mathbf{v}]_i}{\partial X_i} &= \frac{\partial JF^{-1}_{ik}v_k}{\partial X_i}\\ &= \frac{\partial JF^{-1}_{ik}}{\partial X_i}v_k + JF^{-1}_{ik}\frac{\partial v_k}{\partial X_i}\\ &= \underbrace{\frac{\partial JF^{-T}_{ki}}{\partial X_i}}_{0}v_k + JF^{-1}_{ik}\frac{\partial v_k}{\partial X_i}\\ &= J\frac{\partial X_i}{\partial x_k}\frac{\partial v_k}{\partial X_i}\\ &= J\nabla\cdot\mathbf{v}\end{aligned}$$

v

$$\begin{aligned}\frac{\partial[J\mathbf{A}\mathbf{F}^{-T}]_i}{\partial X_i} &= \frac{\partial JA_{ik}F^{-T}_{kj}}{\partial X_j}\\ &= \underbrace{\frac{\partial JF^{-T}_{kj}}{\partial X_j}}_{0}A_{ij} + JF^{-T}_{kj}\frac{\partial A_{ik}}{\partial X_j}\\ &= J\frac{\partial X_j}{\partial x_k}\frac{\partial A_{ik}}{\partial X_j}\\ &= J\nabla\cdot\mathbf{A}\end{aligned}$$

c Requires $\nabla\cdot\mathbf{P} = J\nabla\cdot\sigma$. Above result implies $\mathbf{P} = J\sigma\mathbf{F}^{-T}$.

3 a Shape is a unit square undergoing pure shear in the $\mathbf{e}_1$-direction.

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

b i

$$E = (1/2)(U^T R^T R U - I) = (1/2)(U^T U - I)$$

by orthogonality of the rotation tensor. Therefore the strain tensor depends on the stretch only.

ii Eigenvalues of a tensor/matrix do not change under a rotation.

iii

$$\begin{aligned} E &= (1/2)(U^T U - I) \\ &= (1/2)(\sum_i \lambda_i^2 \mathbf{u}_i \otimes \mathbf{u}_i - \mathbf{I}) \end{aligned}$$

were λ_i and $\mathbf{u}_i$ are the eigenvalues and eigenvectors of $\mathbf{U}$. Note the $\mathbf{U}^T = \mathbf{U}$ and multiplication changes the eigenvalues but not the eigenvectors.

iv The eigenvalues are the 'principal stretches', i.e. the ratio by which a fibre aligned with a particular eigenvector is stretched. The eigenvectors of $\mathbf{U}$ are the stretch directions in the material configuration. The eigenvectors of $\mathbf{U}$ are the stretch directions in the spatial (current) configuration.

v For an incompressible problem, $\det \mathbf{F} = \det(\mathbf{RU}) = \det \mathbf{R} \det \mathbf{U} = 1$. $\det \mathbf{R} = 1$ since $\mathbf{R}$ is a rotation, hence $\det \mathbf{U} = 1$ must hold. This requires that $\lambda_1\lambda_2\lambda_3 = 1$.

vi

$$\begin{aligned} \mathbf{V} &= \mathbf{R}\mathbf{U}\mathbf{R}^T \\ &= \mathbf{R}\left(\sum_i \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i\right)\mathbf{R}^T \\ &= \sum_i \lambda_i \mathbf{R}\mathbf{u}_i \otimes \mathbf{R}\mathbf{u}_i) \\ &= \sum_i \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i), \end{aligned}$$

therefore $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$.

$$\begin{aligned} \mathbf{F} &= \mathbf{R}\mathbf{U} \\ &= \sum_i \lambda_i \mathbf{R}\mathbf{u}_i \otimes \mathbf{u}_i \\ &= \sum_i \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i \end{aligned}$$