EGT3
ENGINEERING TRIPOS PART IIB

Monday 1 May 2023   2 to 3.40

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C9 datasheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A slender elastic cantilever beam has length L, width b and depth $2h_1$, as shown in Fig. 1. There is zero displacement and rotation at the root ($x_1 = 0$). The tip ($x_1 = L$) is unconstrained. The beam is coated by a viscoelastic layer of depth h_2 and width b on both the top and bottom faces. A vertical force F is applied to the tip.

Infinitesimal deformations can be assumed, with the displacement field across all three layers given by

$$\mathbf{u}\left(x_1, x_2\right) = w\left(x_1\right)\, \mathbf{e}_2 - w_{,1}\left(x_1\right) x_2\, \mathbf{e}_1\ .$$

The initial position of a material point is $\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2$, w is the deflection of a point on the beam centre-line (i.e. at $x_2 = 0$) and $w_{,1}$ is the rotation of the cross-section, which is assumed to remain planar.

(a) The elastic portion of the beam has Young's modulus E_1 and Poisson's ratio ν_1. Explain why the elastic strain energy density in this part of the beam is given by

$$U = \frac{1}{2}E_1\varepsilon_{11}^2$$

where $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ are the infinitesimal strain components. [10%]

(b) Neglecting the viscoelastic layers (i.e. the case $h_2 = 0$), and neglecting any body forces, use the method of minimum potential energy to show that the beam deflection must satisfy $w_{,1111} = 0$ for $0 \leq x_1 \leq L$. Write down the boundary conditions required to solve for $w\left(x_1\right)$, although a solution is not required. [40%]

(c) Now include the viscoelastic layers (i.e. $h_2 \neq 0$), which have relaxation modulus $E_r\left(t\right) = E_2 \exp\left(-tE_2/\eta_2\right)$, where t is time and E_2 and η_2 are material constants. Again, body forces can be neglected. It is required that the root curvature $w_{,11}\left(x_1 = 0,\ t\right) = \alpha t$, where α is a constant curvature rate. Derive an expression for the required time-dependent tip force $F\left(t\right)$. [50%]

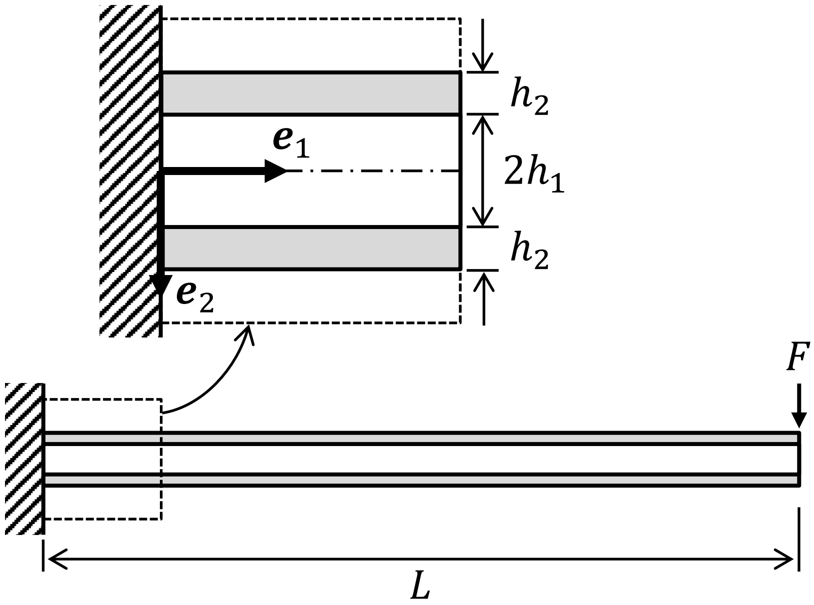

Fig. 1

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2 (a) Using indicial notation, prove the following identities.

(i)

$$\int_V \nabla \times \mathbf{a} \, dV = \oint_S \mathbf{n} \times \mathbf{a} \, dS \, ,$$

where $\mathbf{a}$ is a vector and S is a closed surface, with unit normal $\mathbf{n}$, enclosing a volume V. [10%]

(ii)

$$\nabla a = \frac{\mathbf{a}}{a} \, ,$$

where $a = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ is the magnitude of vector $\mathbf{a}$. [10%]

(iii)

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \, (\nabla \cdot \mathbf{b}) - \mathbf{b} \, (\nabla \cdot \mathbf{a}) + \mathbf{b} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b} \, ,$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors. [20%]

(b) For a deformation map $\mathbf{x} = \phi(\mathbf{X}, t)$, $\mathbf{F} := \nabla_0 \phi(\mathbf{X}, t) = \partial \phi(\mathbf{X}, t)/\partial \mathbf{X}$ is the deformation gradient and $J := \det \mathbf{F}$. Prove the following results:

(i) $\nabla f = \mathbf{F}^{-T} \nabla_0 f$, where f is a scalar. [10%]

(ii) $\nabla \mathbf{u} = (\nabla_0 \mathbf{u}) \mathbf{F}^{-1}$, where $\mathbf{u}$ is a vector. [10%]

(iii) The Piola identity $\nabla_0 \cdot (J\mathbf{F}^{-T}) = \mathbf{0}$, using Nanson's formula $\mathbf{n} ds = J\mathbf{F}^{-T} \mathbf{N} dS$. [10%]

(iv) $\nabla_0 \cdot (J\mathbf{F}^{-1}\mathbf{v}) = J \nabla \cdot \mathbf{v}$, where $\mathbf{v}$ is a vector. [10%]

(v) $\nabla_0 \cdot (J\mathbf{A}\mathbf{F}^{-T}) = J \nabla \cdot \mathbf{A}$, where $\mathbf{A}$ is a second-order tensor. [10%]

(c) The equilibrium equation, neglecting inertia terms, on the reference domain Ω_0 is $-\nabla_0 \cdot \mathbf{P} + \mathbf{b}_0 = \mathbf{0}$, and on the spatial (current) domain Ω is $-\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$. Using identities in (b), give the expression for the nominal stress $\mathbf{P}$ in terms of the Cauchy stress $\boldsymbol{\sigma}$. [10%]

3 (a) Consider a body that undergoes the deformation

$$\mathbf{x} = \phi(\mathbf{X}, t) = \mathbf{X} + \alpha(t)(\mathbf{X} \cdot \mathbf{e}_2)\mathbf{e}_1,$$

where $\alpha > 0$ and $\mathbf{e}_i$ is the canonical unit basis vector in the ith direction. Sketch the deformed shape when $\phi(\mathbf{X}, t)$ is applied to a unit square and compute the Green–Lagrange strain $\mathbf{E} := (1/2)(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. [20%]

(b) A symmetric tensor $\mathbf{A}$ can be expressed as $\mathbf{A} = \sum_i \lambda_i \mathbf{a}_i \otimes \mathbf{a}_i$, where λ_i are the eigenvalues and $\mathbf{a}_i$ are the normalised eigenvectors of $\mathbf{A}$.

The deformation gradient $\mathbf{F}$ admits unique decompositions into stretch and (proper) rotation tensors $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where $\mathbf{U}$ and $\mathbf{V}$ are symmetric with positive eigenvalues and $\mathbf{R}$ is a rotation matrix.

(i) Show that the Green–Lagrange strain is not affected by rotations. [10%]

(ii) Explain why the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$ are the same. [10%]

(iii) Give the Green–Lagrange strain in terms of the eigenvalues and eigenvectors of the tensor $\mathbf{U}$. [10%]

(iv) Give a physical interpretation of the eigenvalues and eigenvectors of $\mathbf{U}$ and $\mathbf{V}$. [20%]

(v) What is the relationship between the eigenvalues of $\mathbf{U}$ for an incompressible problem? [10%]

(vi) Using the decompositions $\mathbf{U} = \sum_i \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$ and $\mathbf{V} = \sum_i \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$, give an expression for the deformation gradient $\mathbf{F}$ in terms of the eigenvalues and eigenvectors of $\mathbf{U}$ and $\mathbf{V}$. Note that $\mathbf{A}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{Au}) \otimes \mathbf{v}$ and $(\mathbf{u} \otimes \mathbf{v})\mathbf{A} = \mathbf{u} \otimes (\mathbf{A}^T\mathbf{v})$. [20%]

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ENGINEERING TRIPOS PART IIB

Module 4C9 Continuum Mechanics

Data sheet

Indicial notation

A repeated index implies summation

$\boldsymbol{a} = a_i \boldsymbol{e}_i \qquad \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ can be written as $c_i = e_{ijk} a_j b_k$

$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ can be written as $A_{ij} = a_i b_j$

Kronecker delta: $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$

Note that $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

Permutation symbol: $e_{ijk} = 1$ when i, j, k are in cyclic order

$e_{ijk} = -1$ when i, j, k are in anti-cyclic order

$e_{ijk} = 0$ when any indices repeat

$e - \delta$ identity: $e_{ijk} e_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$

$\text{grad}\, \phi = \nabla \phi = \phi_{,i} \boldsymbol{e}_i$

$\text{div}\, \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = v_{i,i}$

$\text{curl}\, \boldsymbol{v} = \nabla \times \boldsymbol{v} = e_{ijk} v_{k,j} \boldsymbol{e}_i$

Gauss's theorem (the divergence theorem):

$$\int_V \frac{\partial A_{ij}}{\partial x_j} dV = \oint_S A_{ij} n_j \, dS$$

Stokes's theorem:

$$\int_S e_{ijk} \frac{\partial A_{pk}}{\partial x_j} n_i dS = \oint_C A_{pk} \, dx_k$$

Isotropic linear elasticity

Equilibrium: $\sigma_{ij,j} + b_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji}$

Compatibility: $\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$

Constitutive relationships: $\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

Lame's constants: $\mu = G = \frac{E}{2(1+\nu)} \quad , \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

The strain energy density U is given by: $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$

At equilibrium, the potential energy Π is minimised. Hence, for any small kinematically admissible perturbation δu_i :

$$\delta\Pi = \int_V \delta U dV - \int_S t_i^e \delta u_i dS - \int_V b_i \delta u_i dV = 0$$

Definitions: σ_{ij} is the stress tensor, ε_{ij} is the infinitesimal strain tensor, b_i is the body force vector, t_i^e is the external traction vector and u_i is the displacement vector.

Isotropic linear viscoelasticity

Relaxation modulus, $E_r(t)$:

if $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\sigma(t) = \varepsilon_0 E_r(t)$

Creep compliance, $J_c(t)$:

if $\sigma(t) = \sigma_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\varepsilon(t) = \sigma_0 J_c(t)$

The Laplace transforms of $E_r(t)$ and $J_c(t)$ are related by: $\bar{E}_r(s)\,\bar{J}_c(s) = \frac{1}{s^2}$

Boltzmann superposition principle in 1D:

$$\sigma(t) = \int_0^t \frac{\partial \varepsilon(\tau)}{\partial \tau} E_r(t-\tau)d\tau$$

$$\varepsilon(t) = \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J_c(t-\tau)d\tau$$

Correspondence principle: in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \to s\bar{E}_r(s)$, $\nu \to s\bar{\nu}_r(s)$ (for any time-dependent moduli).