Part II B, 2014

CRIB for 4C9

'Continuum Mechanics'

N.A. Fleck
Principal Assessor

1. (a) (i) $\delta_{ij}\delta_{ik}\delta_{jk} =$

$$\delta_{jk}\delta_{jk} = \delta_{kk} = 3$$

(ii) $\varepsilon_{ijk} a_j a_k = \underset{\sim}{a} \times \underset{\sim}{a} = 0$

(iii) $\varepsilon_{pqs}\varepsilon_{sqr} =$

$\varepsilon_{qsp}\varepsilon_{qrs}$ use $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

$\parallel$

$$\delta_{sr}\delta_{ps} - \delta_{ss}\delta_{pr}$$

$$= \delta_{pr} - 3\delta_{pr} = -2\delta_{pr}$$

(b) $\varepsilon_{pqi} b_i = \varepsilon_{pqi}\left(\frac{1}{2}\varepsilon_{ijk} B_{jk}\right)$

$$= \frac{1}{2}\varepsilon_{ipq}\varepsilon_{ijk} B_{jk}$$

$$= \frac{1}{2}\left(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}\right) B_{jk}$$

$$= \frac{1}{2}\left(B_{pq} - B_{qp}\right)$$

skew so $B_{qp} = -B_{pq}$

$$= \frac{1}{2}\left(B_{pq} + B_{pq}\right) = B_{pq}.$$

(c)

$$\underset{\sim}{V} = \underset{\sim}{b} \times \underset{\sim}{c} \rightarrow V_i = \varepsilon_{ijk} b_j c_k$$

$$\underset{\sim}{w} = \underset{\sim}{a} \times \underset{\sim}{v} \Rightarrow w_p = \varepsilon_{pqi} a_q \varepsilon_{ijk} b_j c_k$$

$$= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}) a_q b_j c_k$$

$$= a_q b_p c_q - a_q b_q c_p$$

$$= (a_q c_q) b_p - (a_q b_q) c_p$$

$$= (\underset{\sim}{a} \cdot \underset{\sim}{c}) \underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{c}$$

(d)

$$\frac{\partial x_i}{\partial X_j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}$$ deformation gradient

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2/2 & A \\ 0 & A & A^2/2 \end{bmatrix}$$ Lagrangian finite strain

for small A

$$\varepsilon_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & A & 0 \end{bmatrix}$$ infinitessimal strain

$\varepsilon_{23} = \varepsilon_{32} = A$

2(a) $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$ for $\sigma_{ij} = -p\delta_{ij}$

Bulk modulus $K = \frac{-p}{\varepsilon_{ii}}$

$\sigma_{kk} = -3p$

$$\varepsilon_{ij} = \frac{1+\nu}{E}(-p\delta_{ij}) - \frac{\nu}{E}\delta_{ij}(-3p)$$

$$= -p\left(\frac{1+\nu}{E}\right)\delta_{ij} + \frac{\nu}{E}\delta_{ij}\,3p$$

$$\varepsilon_{ii} = -3p\left(\frac{1+\nu}{E}\right) + 9p\frac{\nu}{E}$$

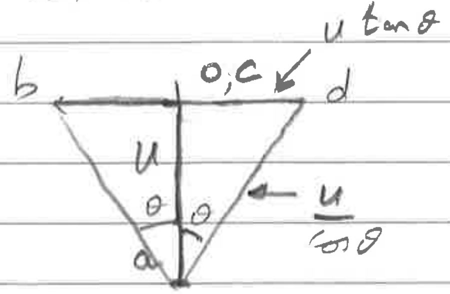
$$= \frac{-3p}{E} + \frac{6p\nu}{E} = \frac{p}{E}(-3+6\nu)$$

$$K = \frac{-p}{\varepsilon_{ii}} = \frac{-p}{p/E\,(-3+6\nu)} = \frac{E}{3(1-2\nu)}$$

(b) The Boussinesq solution is for the elastic problem of a point load on an elastic half-space. z-tractions are zero everywhere except at $r=0$, where a point force P is applied. There is radial symmetry and $u_i = (u_r, 0, u_z)$ and $\sigma_{r\theta} = \sigma_{\theta z} = 0$ (plus symmetric components). The solution is useful because it can be integrated to give solutions for other contact problems such as flat punch indentation.

2. (c)

$$s = \frac{h}{\sin\theta} \qquad \tan\theta = \frac{h}{w}$$



$$Pu = 4k \cdot \frac{h}{\sin\theta} \frac{u}{\cos\theta} \Rightarrow P = \frac{4kh}{\sin\theta\cos\theta}$$

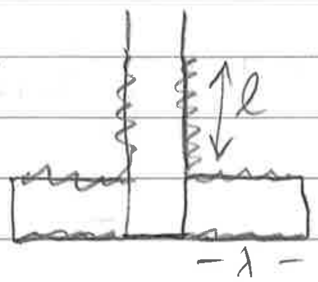
Now $\tan\theta + \dfrac{1}{\tan\theta} = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\cos\theta\sin\theta}$

$$\Rightarrow \quad P = 4kh \cdot \left(x + \frac{1}{x}\right) \qquad x = \tan\theta$$

$$\frac{dP}{dx} = 0 \Rightarrow 1 - \frac{1}{x^2} = 0$$

$$\Rightarrow x = 1 \Rightarrow \theta = \pi/4.$$

So, $\quad P = 4kh \cdot \left(\dfrac{h}{w} + \dfrac{w}{h}\right) \equiv P_f$



$$Pu = P_f \cdot u + 2luk + 4\lambda u \frac{h}{w}$$

$$\Rightarrow \Delta P = P - P_f = 2lk + 4\lambda\frac{h}{w}$$

3. (a) Upper bound : assumes a velocity field & yield. Equilibrium is not satisfied.

Lower bound : assumes an equilibrium stress field & yield is not violated.

Slip line field : satisfies kinematics, equilibrium & yield condition. Hence it is an <u>exact</u> 2D solution.

Note, none of the above methods is unique.

(b) Yield law is $f \equiv \sigma_e - \sigma_Y \leq 0$

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial f}{\partial s_{ij}} \quad \text{by normality.}$$

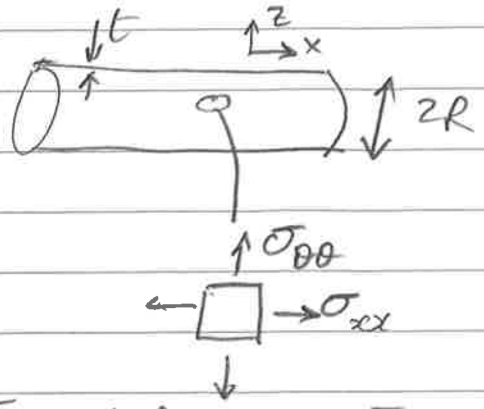
Here, $\dfrac{\partial f}{\partial s_{ij}} = \dfrac{\partial \sigma_e}{\partial s_{ij}}$ Now $\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij}$

$\Rightarrow \quad 2\sigma_e \dfrac{\partial \sigma_e}{\partial s_{ij}} = 3 s_{ij} \quad \Rightarrow \dfrac{\partial \sigma_e}{\partial s_{ij}} = \dfrac{3}{2} \dfrac{s_{ij}}{\sigma_e}$

So $\dot{\varepsilon}_{ij}^{p} \propto s_{ij}$ by normality.

This is the case for $\dot{\varepsilon}_{ij}^{p} = \dfrac{3}{2} \dfrac{s_{ij}}{\sigma_e} \dot{\varepsilon}_e^{p}$

3. (c)



$\sigma_h \cdot t = p \cdot R$

$\Rightarrow \quad \sigma_{\theta\theta} = \dfrac{pR}{t} \qquad \sigma_{xx} = \dfrac{1}{2}\dfrac{pR}{t} = \dfrac{1}{2}\sigma_{\theta\theta}$

$\sigma_{zz} \approx 0$, through thickness.

$\Rightarrow \quad \sigma_m \equiv \dfrac{1}{3}\sigma_{ii} = \dfrac{1}{2}\sigma_{\theta\theta} \quad \Rightarrow \quad S_{xx} = 0$

$S_{\theta\theta} = \sigma_{\theta\theta} - \sigma_m = \dfrac{1}{2}\dfrac{pR}{t} = -S_{zz}$

Flow rule $\Rightarrow \quad \dot{\varepsilon}_{xx} \propto S_{xx} = 0$

So, the tube does not lengthen.

$\dfrac{\dot{t}}{t} = \dot{\varepsilon}_{zz}$ and $\dot{\varepsilon}^p_{zz} = \dfrac{3}{2}\dfrac{S_{zz}}{\sigma_e}\dot{\varepsilon}^p_e$

Now $\sigma_e^2 = \dfrac{3}{2} S_{ij} S_{ij} = \dfrac{3}{2}\left(S_{\theta\theta}^2 + S_{zz}^2\right) = 3 S_{\theta\theta}^2$

$\Rightarrow \quad \sigma_e = \sqrt{3}\, S_{\theta\theta} = \dfrac{\sqrt{3}}{2}\dfrac{pR}{t}$

$\Rightarrow \quad \dot{\varepsilon}^p_{\theta\theta} = -\dfrac{3}{2}\dfrac{S_{\theta\theta}}{\sigma_e}\dot{\varepsilon}^p_e = \dfrac{\sqrt{3}}{2}\dot{\varepsilon}^p_e$

$\dot{\varepsilon}_{zz} = -\dot{\varepsilon}^p_{\theta\theta} = -\dfrac{\sqrt{3}}{2}\dot{\varepsilon}^p_e$

(d) $\dfrac{\dot{t}}{t} = -\dfrac{\sqrt{3}}{2}\dot{\varepsilon}^p_e \quad \Rightarrow \quad \underline{\dfrac{t}{t_0} = \exp\left(-\dfrac{\sqrt{3}}{2}\varepsilon_e\right)}$

3(d) contd.

$$\sigma_e = \frac{\sqrt{3}}{2} \frac{pR}{t}$$

$$2\pi R t = 2\pi R_0 t_0 \quad \Rightarrow \quad R = \frac{R_0 t_0}{t}$$

$$\Rightarrow \quad \sigma_e = \frac{\sqrt{3}}{2} p \frac{R_0 t_0}{t^2} = \frac{\sqrt{3}}{2} p \frac{R_0}{t_0} \left(\frac{t_0}{t}\right)^2$$

Also, $\sigma_e = A \varepsilon_e^N \quad \Rightarrow \quad \varepsilon_e = \left(\frac{\sigma_e}{A}\right)^{1/N}$

$$\ln\left(\frac{t}{t_0}\right) = -\frac{\sqrt{3}}{2} \varepsilon_e = -\frac{\sqrt{3}}{2} \left(\frac{\sigma_e}{A}\right)^{1/N}$$

$$\Rightarrow \quad \ln\left(\frac{t_0}{t}\right) = \frac{\sqrt{3}}{2} A^{-1/N} \left(\frac{\sqrt{3}}{2} \frac{R_0}{t_0} p\right)^{1/N} \left(\frac{t_0}{t}\right)^{2/N}$$

$$\Rightarrow \quad \left(\frac{t}{t_0}\right)^{2/N} \ln\left(\frac{t_0}{t}\right) = \frac{\sqrt{3}}{2} A^{-1/N} \left(\frac{\sqrt{3}}{2} \frac{R_0}{t_0}\right)^{1/N} p^{1/N}$$