Examiner's comments - 4C9

Q1 Strain energy density
This question was attempted by all candidates, and nearly all answers were very good.

Q2 Index notation and deformation gradient
This question was attempted by all candidates. The index manipulation parts of this question were generally well answered. A common mistake was to write expressions in which an index appeared more than twice, which is not possible. On part (b), relatively few candidates expanded the expression for the Green-Lagrange strain in a form that included the displacement, which is needed to show the equivalence with the linearised strain when gradients are small. On part (c), some candidates omitted the '1' on the diagonal of the deformation gradient for the z-direction and mistakenly set it to zero. A small number of candidates asserted that det(F) = 0 in the case of no volume change, whereas it should be det(F) = 1. Few candidates demonstrated that Nanson's formula holds by simply examining the normal vector on a face of the cube before and after motion.

Q3 Nonlinear continuum mechanics
This question was not attempted by any candidates.

Q1

(a) Displacements, small strains and rotations: point at $\underline{x} = x_1 \underline{e}_1 + R\underline{e}_3$

$\underline{u} = u_2(x_1)\,\underline{e}_2$ (i.e. rotation about $\underline{e}_1$ axis)

Strains: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\frac{\partial u_2}{\partial x_1}, \quad \varepsilon_{11} = \varepsilon_{22} = 0$$

Strain energy (linear elastic)

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{12}\varepsilon_{12} + \frac{1}{2}\sigma_{21}\varepsilon_{21} = \sigma_{12}\varepsilon_{12} \quad \text{(plane stress)}$$

Constitutive law: $\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$

$$\therefore U = \frac{E}{1+\nu}\varepsilon_{12}^2 = \frac{E}{4(1+\nu)}(u_{2,1})^2$$

(b) Potential energy:

$$\Pi = \int_V U(x_1)\,dV - \int_S t_i^e u_i\,dS$$

External work term: $\int_S t_i^e u_i\,dS = T\,\theta(L) = T\frac{u_2(L)}{R}$

rotation at $x_1 = L$

θ in terms of u_2, for a point at $\underline{x}$

Sub. for $U(x_1)$:

$$\Pi = \int_V \frac{E(x_1)}{4(1+\nu)}(u_{2,1})^2\,dV - T\frac{u_2(L)}{R}$$

Given deformation is constant around tube circumference at a given x_1:

$$\Pi = \int_0^L \frac{E(x_1)}{4(1+\nu)}(u_{2,1})^2\left[2\pi R\,t(x_1)\right]dx_1 - T\frac{u_2(L)}{R}$$

$$\therefore \Pi = \int_0^L \frac{\pi R\,E(x_1)\,t(x_1)}{2(1+\nu)}(u_{2,1})^2\,dx_1 - T\frac{u_2(L)}{R}$$

(c) Variation in Π:

$$\delta\Pi = \int_V \frac{\partial U}{\partial u_{2,1}} \delta u_{2,1} \, dV - T \frac{\delta u_2(L)}{R}$$

$$= \int_0^L \frac{\pi R}{1+\nu} \underbrace{E(x_1)\, t(x_1)\, (u_{2,1})}_{\text{note: all a function of } x_1} \overset{\delta u_{2,1}}{\uparrow} dx_1 - T \frac{\delta u_2(L)}{R}$$

Integrate by parts:

$$\delta\Pi = \left[\frac{\pi R}{1+\nu} E(x_1)\, t(x_1)\, u_{2,1}(x_1)\, \delta u_2\right]_0^L$$

$$- \int_0^L \frac{\pi R}{1+\nu} \frac{d}{dx_1}\left[E(x_1)\, t(x_1)\, u_{2,1}(x_1)\right] \delta u_2 \, dx_1 - T \frac{\delta u_2(L)}{R}$$

Minimum PE: $\delta\Pi = 0$ for any kinematically admissible $\delta u_2(x_1)$

$\therefore$ Along the length: $\frac{d}{dx_1}\left[E(x_1)\, t(x_1)\, u_{2,1}(x_1)\right] = 0$

$$\therefore \quad E(x_1)\, t(x_1)\, u_{2,1}(x_1) = C_1 \quad (1) \leftarrow \text{constant}$$

$\rightarrow$ governing equation for $u_2(x_1)$

At $x=0$: $E(0)\, t(0)\, u_{2,1}(0)\, \delta u_2(0) = 0$

$\uparrow = 0$, boundary condition given: $u_2(0) = 0$ (2)

At $x=L$: $\left[\frac{\pi R}{1+\nu} E(L)\, t(L)\, u_{2,1}(L) - \frac{T}{R}\right] \delta u_2(L) = 0$

$$\Rightarrow \quad \frac{\pi R}{1+\nu} E(L)\, t(L)\, u_{2,1}(L) - \frac{T}{R} = 0$$

$$\therefore \quad u_{2,1}(L) = \frac{T(1+\nu)}{\pi R^2 E(L)\, t(L)} \quad (3) \text{ (B.C. on } u_2(L))$$

(d) Step 1 : find constant C_1 in (1), using B.C. (3) and measured values of $t(L)$, $E(L)$

Step 2 : obtain $E(x_1)$ and $t(x_1)$ from measurements, then integrate (1), using B.C. (2)

$$u_2(x_1) = \int \frac{C_1}{E(x_1)\, t(x_1)}\, dx_1$$

Step 3 : rotation at end of tube $\theta(L) = \dfrac{u_2(L)}{R}$

$\boxed{Q2}$

(a) (i) If $\phi(\underline{x})$ is a scalar field:

$$\nabla \times (\nabla \phi) = e_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_k}\right) = e_{ijk} \underbrace{\frac{\partial^2 \phi}{\partial x_j \partial x_k}}_{\text{symmetric}: \ \phi_{,jk} = \phi_{,kj}}$$

Given: $e_{ijk} = -e_{ikj} \Rightarrow e_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0$, given

(ii) Rotated basis vectors: $\hat{\underline{e}}_i = R_{ij} \underline{e}_j$

$$\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = (R_{ik} \underline{e}_k)(R_{jp} \underline{e}_p) = R_{ik} R_{jp} \delta_{kp}$$
$$= R_{ip} R_{jp}$$

Also: $\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = \delta_{ij}$ $\therefore$ $R_{ip} R_{jp} = \delta_{ij}$ ①

i.e. $\underline{R} \cdot \underline{R}^T = \underline{I}$

Using $(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T$, and $\underline{I} = \underline{I}^T$, $\Rightarrow \underline{R}^T \cdot \underline{R} = \underline{I}$

i.e. $R_{pi} R_{pj} = \delta_{ij}$ ②

To prove using indicial notation, premultiply ① by $\underline{R}^T$:

$$R_{ik}(R_{ip} R_{jp}) = R_{ik} \delta_{ij}$$

Rearranging terms $(\times \delta_{kp})$:
(to get R_{jk} on LHS)

$$(R_{ik} R_{ip} R_{jp}) \delta_{kp} = (\overbrace{R_{ik} \delta_{ij}}^{R_{jk}}) \delta_{kp}$$

$$(R_{ik} R_{ip}) R_{jk} = \delta_{kp} R_{jk}$$

$\therefore$ $R_{ik} R_{ip} = \delta_{kp} \rightarrow$ ②

(iii) If: $\underline{A} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j = \hat{A}_{ij}\, \hat{\underline{e}}_i \otimes \hat{\underline{e}}_j$

And: $\hat{\underline{e}}_i = R_{ij}\, \underline{e}_j \Rightarrow R_{ik}\, \hat{\underline{e}}_i = R_{ik} R_{ij}\, \underline{e}_j$ $\quad (\times\, \underline{R}^T)$

$= \delta_{kj}\, \underline{e}_j$ $\quad$ (using ②)

$= \underline{e}_k$

Sub for $\underline{e}_i$: $A_{ij}\, \underline{e}_i \otimes \underline{e}_j = A_{ij} \left(R_{ki}\, \hat{\underline{e}}_k \right) \otimes \left(R_{pj}\, \hat{\underline{e}}_p \right)$

$= \left(A_{ij} R_{ki} R_{pj} \right) \hat{\underline{e}}_k \otimes \hat{\underline{e}}_p$

$\therefore\ \underline{\hat{A}_{ij} = A_{pq} R_{ip} R_{jq}}$ $\quad$ (after change of variables)

2 b) $E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$

$F_{ij} = \delta_{ij} + u_{i,j}$

$E_{ij} = \frac{1}{2}\left[(\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) - \delta_{ij}\right)$

$= \frac{1}{2}\left[\delta_{ki}\delta_{kj} + \delta_{ki}u_{k,j} + u_{k,i}\delta_{kj} + u_{k,i}u_{k,j} - \delta_{ij}\right]$

$= \frac{1}{2}\left[\cancel{\delta_{ij}} + u_{i,j} + u_{j,i} + + u_{k,i}u_{k,i} - \cancel{\delta_{ij}}\right)$

For linearised kinematics $u_{k,i}u_{k,i} \to 0$

$\Rightarrow \frac{1}{2}(u_{i,j} + u_{j,i})$

c) i) $\underline{y} = \underline{\underline{R}}\,\underline{X} + \underline{c}$, $\underline{\underline{R}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underline{c}$ = constant

$\underline{\underline{F}} = \frac{\partial \underline{y}}{\partial \underline{X}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

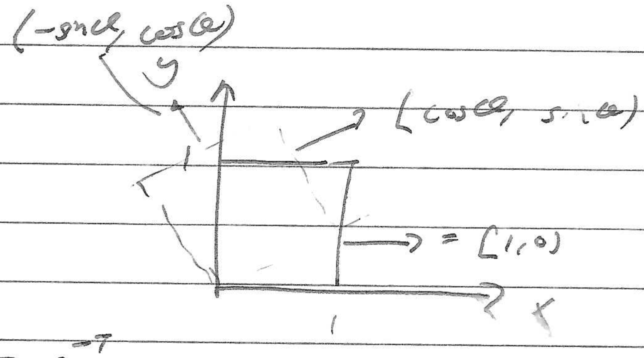
ii) $\det \underline{\underline{F}} = 1$ because motion is rigid-body

iv) ~~iii)~~ $\underline{\underline{F}}^T\underline{\underline{F}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \cancel{\neq}\ \underline{\underline{I}}$

$\frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}) = \underline{\underline{0}}$ (motion is rigid body $\to$ zero strain)

2 c) iii)



$$\underline{ds} = J F^{-T} \underline{dS}$$

$$J \underset{\rightarrow}{F}^{-T} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-T} \qquad (J=1)$$

$$= \frac{1}{1}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \\ & \end{bmatrix}^{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{F}}^{-T}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \checkmark$$

$$\underline{\underline{F}}^{-T}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \checkmark$$

No change in normal in the X_3 direction

3 a) i) SVD: $\underline{\underline{F}} = \underline{\underline{W}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}}^T$, all 3×3

$$= \underbrace{\underline{\underline{W}}\,\underline{\underline{Z}}^T}_{\underline{\underline{R}}}\underbrace{\underline{\underline{Z}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}}^T}_{\underline{\underline{U}}}$$

$$= \underline{\underline{R}}\,\underline{\underline{U}}, \quad \underline{\underline{R}} = \underbrace{\underline{\underline{W}}\,\underline{\underline{Z}}^T}_{\text{product of orthog. tensor.}}, \quad \underline{\underline{U}} = \underline{\underline{Z}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}}^T \text{ (symmetric)}$$

$$\underline{\underline{F}} = \underbrace{\underline{\underline{W}}\,\underline{\underline{\Sigma}}\,\underline{\underline{W}}^T}_{\underline{\underline{V}}}\underbrace{\underline{\underline{W}}\,\underline{\underline{Z}}^T}_{\underline{\underline{R}}}$$

symmetric

ii) $\underline{\underline{U}} = \underline{\underline{Z}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}}^T$, $\underline{\underline{U}}^T = (\underline{\underline{Z}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}}^T)^T$

$= \underline{\underline{Z}}\,\underline{\underline{\Sigma}}\,\underline{\underline{Z}} \rightarrow$ symmetric.

$\underline{\underline{\Sigma}}$ are the singular values, i.e. square root of eigenvalues of $\underline{\underline{F}}^T\underline{\underline{F}}$. Singular values are always positive.

iii) Take right Cauchy-Green tensor

$$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$$

$$= \underline{\underline{U}}^T\underbrace{\underline{\underline{R}}^T\underline{\underline{R}}}_{=\underline{\underline{I}}}\underline{\underline{U}}$$

$\rightarrow$ strain tensors cannot depend on rotations. Polar decomposition allows us to show/prove this.

3 (b) Multiply by $\dot{\underline{\phi}}$ and integrate over reference domain

$$\int_{\Omega_0} \rho_0 \dot{\underline{\phi}} \cdot \ddot{\underline{\phi}} \, dX = \int_{\Omega_0} \dot{\underline{\phi}} \cdot (\nabla \cdot \underline{\underline{P}}) \, dX + \int_{\Omega_0} \rho_0 \dot{\underline{\phi}} \cdot \underline{b}_m \, dX$$

Integrate by parts,

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho_0 (\dot{\underline{\phi}} \cdot \dot{\underline{\phi}}) \, dX + \int_{\Omega_0} \underline{\underline{P}} : \overbrace{(\nabla \dot{\underline{\phi}})}^{= \dot{\underline{\underline{F}}}} \, dX$$

$$= \int_{\partial\Omega_0} \underline{\underline{P}} \underline{N} \cdot \dot{\underline{\phi}} \, ds + \int_{\Omega} \rho_0 \dot{\underline{\phi}} \cdot \underline{b}_m \, dX$$

Stress power term: $\int_{\Omega_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dX \Rightarrow$ conjugate

c) The stored-energy must be invariant under rotations.

Take variations (derivative w.r.t. each variable)

3 c) $$D_\phi \Pi = \int_{\Omega_0} \underline{\underline{P}} : \nabla_0 \delta \underline{\phi} \, dV - \frac{\partial \Pi}{\partial \phi} \cdot \delta \phi = 0 \quad (1)$$

$$D_F \Pi = \int \frac{\partial \psi}{\partial F} : \delta \underline{\underline{F}} \, dV \mp \int \underline{\underline{P}} : \delta \underline{\underline{F}} \, dV = 0 \quad (2)$$

$$D_P \Pi = \int \delta P : (\nabla_0 \phi - \underline{\underline{F}}) \, dV = 0 \quad (3)$$

From (3) $\Rightarrow$ $F = \nabla_0 \underline{\phi}$ (defn of deformation gradient)

From (2) $\underline{\underline{P}} = \frac{\partial \psi}{\partial \underline{\underline{F}}}$ (defn. of Piola stress)

Eq (1) $\rightarrow$ $+ \nabla_0 \cdot \underline{\underline{P}} - \partial \Pi / \partial \underline{\phi} = 0$

(equilibrium eqn. on referen domain).