Version GNW/4

EGT3
ENGINEERING TRIPOS PART IIB

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Monday 3 May 2021 1.30 to 3.10

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**Module 4C9**

**CONTINUUM MECHANICS**

*Answer not more than **two** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

**STATIONERY REQUIREMENTS**
Write on single-sided paper.

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed.
Attachment: 4C9 datasheet (2 pages).
You are allowed access to the electronic version of the Engineering Data Books.

**10 minutes reading time is allowed for this paper at the start of the exam.**

**The time taken for scanning/uploading answers is 15 minutes.**

**Your script is to be uploaded as a single consolidated pdf containing all answers.**

1 A thin-walled circular tube of length $L$, radius $R$ and wall thickness $t$ is loaded by a torque $T$ as shown in Fig. 1. The tube is constrained such that the displacement $\mathbf{u} = 0$ at $x_1 = 0$. The initial position $\mathbf{x}$ and displacement $\mathbf{u}$ of a point on the tube wall (assuming plane stress, and infinitesimal strains and rotations) are given by

$$\mathbf{x} = x_1 \mathbf{e}_1 + R\mathbf{e}_3, \quad \mathbf{u} = u_2\left(x_1\right)\mathbf{e}_2.$$

Due to the symmetry, this can describe any material point following a rotation of the orthonomal basis vectors $\left(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\right)$ about the $\mathbf{e}_1$ direction.

The tube is 3D-printed from a linear elastic material. Due to printing defects, the Young's modulus and the wall thickness both vary with axial position, i.e. $E\left(x_1\right)$ and $t\left(x_1\right)$. At a given $x_1$, these properties are constant around the circumference of the tube. The Poisson ratio $\nu$ is constant everywhere.

(a) Show that the strain energy density $U\left(x_1\right)$ is given by:

$$U = \frac{E\left(x_1\right)}{4\left(1+\nu\right)}\left(u_{2,1}\right)^2.$$

[20%]

(b) Show that the potential energy $\Pi$ is given by:

$$\Pi = \int_0^L \frac{\pi R}{2\left(1+\nu\right)} E\left(x_1\right) t\left(x_1\right)\left(u_{2,1}\right)^2 dx_1 - \frac{T}{R} u_2\left(L\right).$$

[20%]

(c) Using the method of minimum potential energy, derive the governing equation for $u_2\left(x_1\right)$ along the length of the tube, and the associated boundary conditions. [50%]

(d) Measurements provide data for $E\left(x_1\right)$ and $t\left(x_1\right)$. Explain, without further calculation, how an expression for the rotation of the tube at $x_1 = L$ would be obtained. [10%]

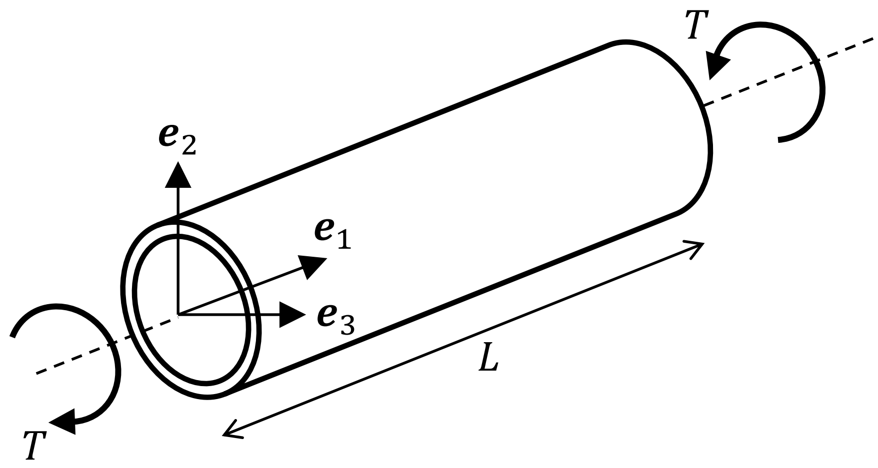


Fig. 1

2 (a) The position vector $\mathbf{x}$ is defined using orthonormal basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Use index notation to answer the following.

(i) If $\phi(\mathbf{x})$ is a scalar field, show that $\nabla \times (\nabla\phi) = 0$. [10%]

(ii) For rotated orthonormal basis vectors $\hat{\mathbf{e}}_i = R_{ij}\mathbf{e}_j$, show that

$$R_{ip}R_{jp} = R_{pi}R_{pj} = \delta_{ij}.$$

[20%]

(iii) For the tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \hat{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, derive an expression for the relationship between components $A_{ij}$, $\hat{A}_{ij}$, and $R_{ij}$. [20%]

(b) The Green–Lagrange strain is given by $\mathbf{E} := (1/2)(\mathbf{F}^{\top}\mathbf{F} - \mathbf{I})$, where $\mathbf{F}$ is the deformation gradient and $\mathbf{I}$ is the identity tensor. Express the Green–Lagrange strain using index notation, and use this to show that the Green–Lagrange strain reduces to the linearised strain measure for linearised kinematics. [10%]

(c) Consider a body in 3D that undergoes an arbitrary translation, plus a rotation about the $X_3$ axis.

(i) Compute the deformation gradient $\mathbf{F}$. [10%]

(ii) Without calculation, give $\det \mathbf{F}$. Explain your reasoning. [5%]

(iii) By considering an axis-aligned unit cube, demonstrate that $\mathrm{d}\mathbf{s} = J\mathbf{F}^{-\top}\,\mathrm{d}\mathbf{S}$ (Nanson's formula) holds. [20%]

(iv) Compute the Green–Lagrange strain. [5%]

3 (a) The deformation gradient $\mathbf{F}$ can be decomposed as:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

where $\mathbf{R}$ is a rotation, and $\mathbf{U}$ and $\mathbf{V}$ are symmetric. This is known as the polar decomposition.

(i) Using the singular value decomposition of a matrix $\mathbf{A} = \mathbf{W}\Sigma\mathbf{Z}^{\top}$, where $\mathbf{W}$ and $\mathbf{Z}$ are orthogonal matrices and $\Sigma$ holds the singular values of $\mathbf{A}$, show that the polar decomposition of the deformation gradient exists. [30%]

(ii) Show that the stretch tensor $\mathbf{U}$ is symmetric and that all eigenvalues of $\mathbf{U}$ are positive. Comment on the physical significance of the eigenvalues being positive. [20%]

(iii) Comment on the significance of the polar decomposition for the definition of strain tensors. [10%]

(b) Starting from the balance of momentum in the material configuration

$$\rho_0 \ddot{\boldsymbol{\phi}} = \nabla_X \cdot \mathbf{P} + \rho_0 \mathbf{b}_m,$$

where $\rho_0$ is the reference density, $\boldsymbol{\phi}$ is the deformation map, $\mathbf{P}$ is the first Piola–Kirchhoff stress and $\mathbf{b}_m$ is the material body force, show that $\mathbf{P}$ is work-conjgate to $\dot{\mathbf{F}}$. [10%]

(c) The Hu–Washizu variational principle involves a total potential energy functional of the form

$$\Pi(\boldsymbol{\phi}, \mathbf{F}, \mathbf{P}) := \int_{\Omega_0} \psi(\mathbf{F})\, \mathrm{d}V + \int_{\Omega_0} \mathbf{P} : (\nabla_X \boldsymbol{\phi} - \mathbf{F})\, \mathrm{d}V - \Pi_{\text{external}}(\boldsymbol{\phi}),$$

where $\boldsymbol{\phi}$, $\mathbf{F}$ and $\mathbf{P}$ are treated as independent variables and $\Pi_{\text{external}}$ is associated with loadings. If equilibrium corresponds to a stationary point of $\Pi$, give any relationships that must exist between $\psi$, $\boldsymbol{\phi}$, $\mathbf{F}$ and $\mathbf{P}$ at equilibrium. [30%]

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## ENGINEERING TRIPOS PART IIB

## Module 4C9 Continuum Mechanics

## Data sheet

### Indicial notation

A repeated index implies summation

$\boldsymbol{a} = a_i \boldsymbol{e}_i \qquad \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ can be written as $c_i = e_{ijk} a_j b_k$

$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ can be written as $A_{ij} = a_i b_j$

Kronecker delta: $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$

Note that $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

Permutation symbol: $e_{ijk} = 1$ when $i, j, k$ are in cyclic order

$e_{ijk} = -1$ when $i, j, k$ are in anti-cyclic order

$e_{ijk} = 0$ when any indices repeat

$e - \delta$ identity: $e_{ijk} e_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$

$\text{grad}\, \phi = \nabla \phi = \phi_{,i} \boldsymbol{e}_i$

$\text{div}\, \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = v_{i,i}$

$\text{curl}\, \boldsymbol{v} = \nabla \times \boldsymbol{v} = e_{ijk} v_{k,j} \boldsymbol{e}_i$

Gauss's theorem (the divergence theorem):

$$\int_V \frac{\partial A_{ij}}{\partial x_j} dV = \oint_S A_{ij} n_j \, dS$$

Stokes's theorem:

$$\int_S e_{ijk} \frac{\partial A_{pk}}{\partial x_j} n_i dS = \oint_C A_{pk} \; dx_k$$

**Isotropic linear elasticity**

Equilibrium: $\sigma_{ij,j} + b_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji}$

Compatibility: $\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$

Constitutive relationships: $\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

Lame's constants: $\mu = G = \frac{E}{2(1+\nu)} \quad , \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

The strain energy density $U$ is given by: $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$

At equilibrium, the potential energy Π is minimised. Hence, for any small kinematically admissible perturbation $\delta u_i$ :

$$\delta\Pi = \int_V \delta U dV - \int_S t_i^e \delta u_i dS - \int_V b_i \delta u_i dV = 0$$

Definitions: $\sigma_{ij}$ is the stress tensor, $\varepsilon_{ij}$ is the infinitesimal strain tensor, $b_i$ is the body force vector, $t_i^e$ is the external traction vector and $u_i$ is the displacement vector.

**Isotropic linear viscoelasticity**

Relaxation modulus, $E_r(t)$:

if $\varepsilon(t) = \varepsilon_0 H(t)$ , where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$ , then $\sigma(t) = \varepsilon_0 E_r(t)$

Creep compliance, $J_c(t)$:

if $\sigma(t) = \sigma_0 H(t)$ , where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$ , then $\varepsilon(t) = \sigma_0 J_c(t)$

The Laplace transforms of $E_r(t)$ and $J_c(t)$ are related by: $\bar{E}_r(s)\,\bar{J}_c(s) = \frac{1}{s^2}$

Boltzmann superposition principle in 1D:

$$\sigma(t) = \int_0^t \frac{\partial \varepsilon(\tau)}{\partial \tau} E_r(t-\tau) d\tau$$

$$\varepsilon(t) = \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J_c(t-\tau) d\tau$$

Correspondence principle: in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \to s\bar{E}_r(s)$ , $\nu \to s\bar{\nu}_r(s)$ (for any time-dependent moduli).