EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 27 April 2022   2 to 3.40

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F1 Formulae sheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The position of a mass is to be controlled through a linear spring. The (normalised) transfer function relating the position of the mass to the free end of the spring is given by

$$G(s) = \frac{1}{s^2 + 1}.$$

(a) A proportional-integral-derivative controller with negative feedback is proposed for $G(s)$ of the form:

$$K(s) = \frac{k_i}{s} + k_p + sk_d.$$

Use the Routh-Hurwitz criterion to determine necessary and sufficient conditions for closed-loop stability treating the two cases separately: (i) $k_i = 0$, (ii) $k_i \neq 0$. [20%]

(b) A proportional-plus-derivative controller in the form $kC(s)$ is selected with $C(s) = s + 1$. Closed-loop stability is to be assessed using a Nyquist diagram of $L_1(s) = C(s)G(s)$.

(i) Sketch an s-plane contour with any necessary imaginary axis indentations along which $G(s)$ will be evaluated. [5%]

(ii) Sketch the complete Nyquist diagram of $L_1(s)$ paying close attention to the image of any semi-circular indentations of the contour in Part (b)(i). [The locus of $L_1(j\omega)$ for $\omega > 0$ is shown in a finite part of the complex plane in Fig. 1.] [10%]

(iii) Determine the number of closed-loop poles with $\text{Re}(s) > 0$ for each real k. [5%]

(c) Repeat Part (b) for a proportional-plus-integral controller with $C(s) = \frac{1}{s} + 1$. [The locus of $L_1(j\omega)$ for $\omega > 0$ is shown in a finite part of the complex plane in Fig. 2.] [20%]

(d) Let $S(s) = (1 + G(s)K(s))^{-1}$ denote the sensitivity function for an internally stabilising controller $K(s)$ of bounded high frequency gain.

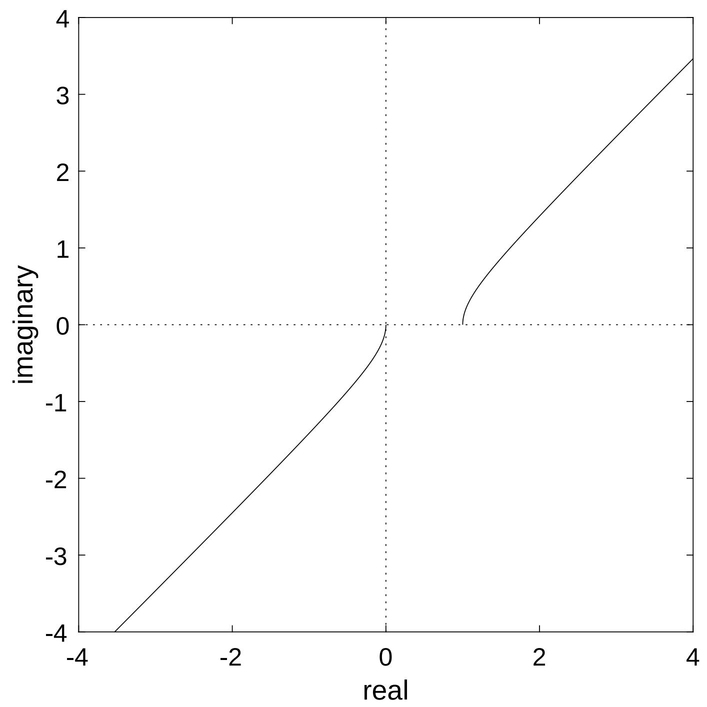
(i) Show that $S(j) = 0$ and $S(\infty) = 1$. [10%]

(ii) Explain why there must be a frequency ω_0 such that $|S(j\omega_0)| > 1$. [You may state without proof any results you use.] [15%]

(iii) By considering the function

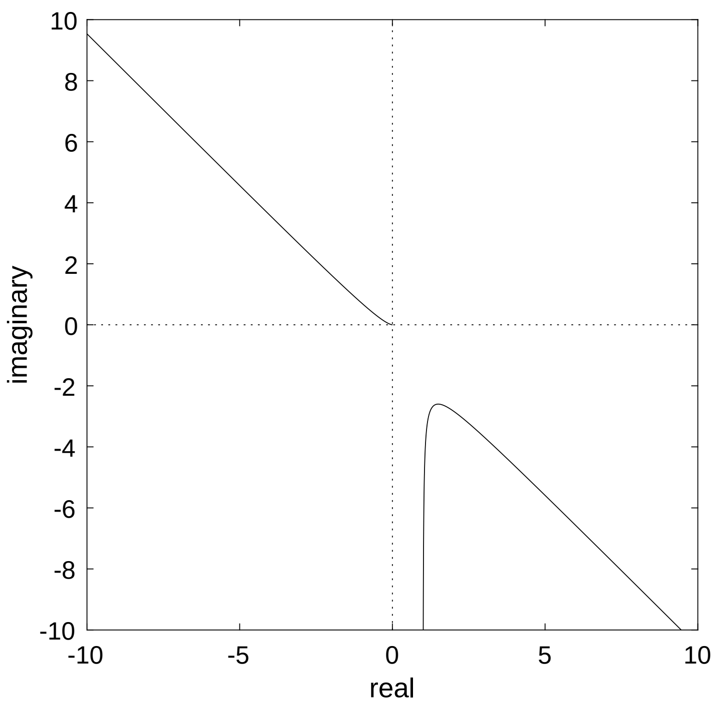
$$S(s) = \frac{s^2 + 1}{(s + 1)^2}$$

show that Part (d)(ii) no longer holds if the high frequency gain requirement on $K(s)$ is removed. [15%]



imaginary
real
4
3
2
1
0
-1
-2
-3
-4
-4
-2
0
2
4

Fig. 1



imaginary
real
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-5
0
5
10

Fig. 2

2 In the linear (non-ideal) operational amplifier circuit of Fig. 3, $Z_1(s)$ and $Z_2(s)$ are circuit impedances which relate the Laplace transforms of the voltage across to current through a circuit element or network, v_1, v_2 are input and output voltages and v is the voltage at the inverting input of the op amp.

(a) Assuming that the currents into the input terminals of the op amp are negligible show that

$$Z_1\bar{v}_2 + Z_2\bar{v}_1 = (Z_1 + Z_2)\bar{v}$$

where $\bar{v}$ denotes the Laplace transform of v etc. [10%]

(b) Suppose that the op amp gain is determined by a transfer function $G(s)$, i.e. $\bar{v}_2 = -G\bar{v}$. Show that the op amp circuit can be represented by the block diagram of Fig. 4. [15%]

(c) Suppose $Z_1 = 1$ and $Z_2 = 6$ and that

$$G(s) = \frac{7000}{10s + 1}.$$

Sketch the Bode diagram of the transfer function relating $-\bar{v}_2$ to $\bar{v}_1$. [15%]

(d) Variations in the op amp gain suggest that it should be modelled with multiplicative uncertainty as: $G_1 = G(1 + \Delta)$ where $|\Delta(j\omega)| \leq h(\omega)$ for all ω. Determine a necessary and sufficient condition for robust stability. You may assume the Small Gain Theorem. [15%]

(e) For Z_1, Z_2 and G as in Part (c) and

$$h(\omega) = \left|\frac{j\omega + 10}{j\omega + 100}\right|$$

show that the op amp circuit is robustly stable. [15%]

(f) Take $G(s)$ as in Part (c), $Z_1 = 1$ and suppose the impedance Z_2 takes the form of a lead compensator:

$$Z_2(s) = \frac{20s + 400}{s + 300}.$$

(i) Show that $Z_1/(Z_1 + Z_2)$ takes the form of a lag compensator. [10%]

(ii) Show with reference to the phase of the return ratio of the feedback loop in Fig. 4, or otherwise, that the nominal op amp circuit is stable. [10%]

(iii) By considering the frequency $s = j100$ show that the op amp circuit is not robustly stable with $h(\omega)$ as in Part (e). [10%]

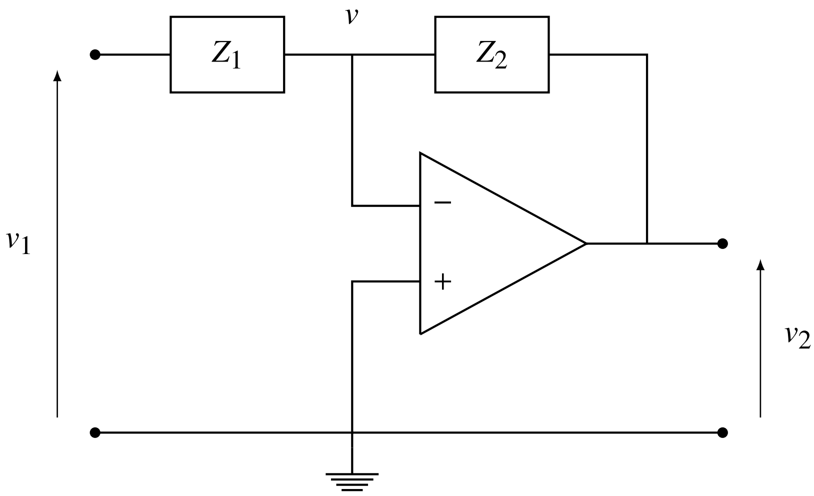


Fig. 3

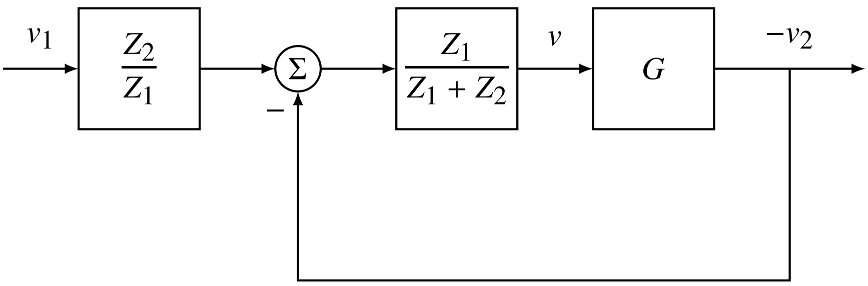


Fig. 4

 (TURN OVER

3 A controller is to be designed for an inverted pendulum on a cart. The transfer-function relating cart velocity to force applied to the cart takes the (scaled) form:

$$G(s) = \frac{100(s^2 - 1)}{s(s^2 - 100)}.$$

(a) (i) Express the transfer-function in the form

$$G(s) = G_m(s)B_p(s)B_z(s)$$

where $B_p(s)$ is a pole-type all-pass function with $B_p(0) = 1$, $B_z(s)$ is a zero-type all-pass function with $B_z(0) = 1$, and $G_m(s)$ has no poles or zeros with $\mathrm{Re}(s) > 0$. [10%]

(ii) Comment briefly on any limitations that may be experienced in the design of a controller for $G(s)$. [15%]

(b) (i) By considering the root-locus of $G(s)$, or otherwise, explain why a stabilising controller for $G(s)$ must contain a right half-plane pole. [10%]

(ii) By considering the real axis portions of the root-locus for $\mathrm{Re}(s) > 0$ explain why a controller with a single right half-plane pole and no right half-plane zeros is unable to stabilise $G(s)$. [15%]

(c) (i) Sketch the root-locus diagram of

$$G_1(s) = \frac{(s - 1)^2}{s(s - 10)^2}$$

and hence verify that this plant can be stabilised by proportional gain feedback. [Hint: the breakaway points are: -5, -2, 1, 10.] [15%]

(ii) Find the value of feedback gain for which there is a double pole at $s = -2$. [10%]

(d) (i) Use Part (c) to write down a stabilising controller $K(s)$ for $G(s)$. [Hint: left half-plane pole zero cancellations between $G(s)$ and $K(s)$ are allowed.] [10%]

(ii) Sketch the Bode diagram of $G(s)K(s)$. [15%]

END OF PAPER

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

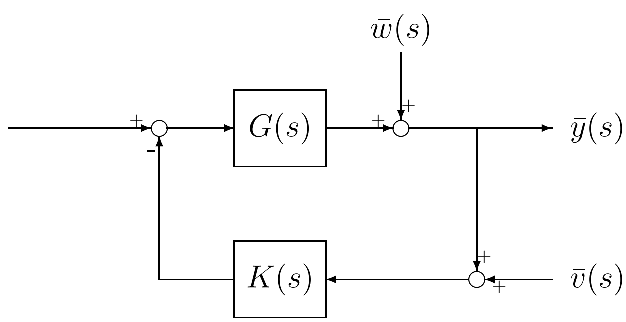
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\,dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

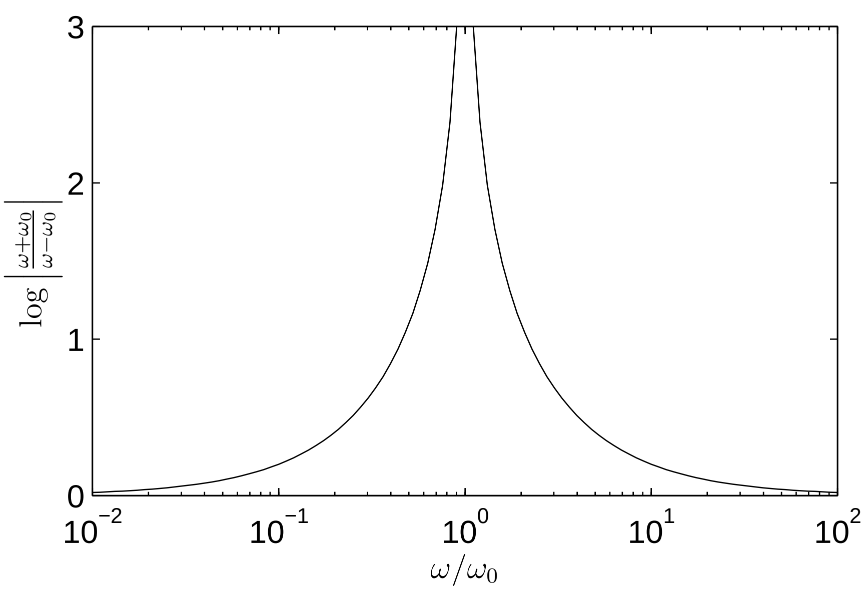


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v)|}{dv}\right|_{v=0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^v\big)\right| dv}$$

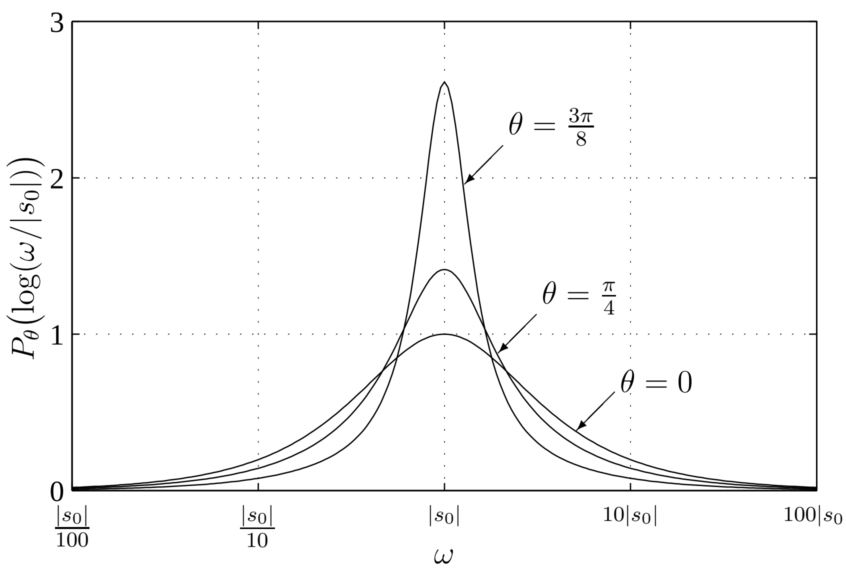
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith

November 2021

Engineering Tripos Part IIB

2022

Paper 4F1: Control System Design

Answers

1(b)(iii) Closed-loop stable for $k > 0$; 2 RHP poles for $-1 < k < 0$; 1 RHP pole for $k < -1$.

1(c)(iii) 2 RHP poles for $k > 0$; 1 RHP pole for $k < 0$.

3(a)(i)

$$G(s) = G_m(s)B_p(s)B_z(s) = \frac{100(1+s)^2}{s(10+s)^2}\ \frac{10+s}{10-s}\ \frac{1-s}{1+s}$$

(c)(ii) Required feedback gain is $k = 32$.

(d)(i)

$$K(s) = 0.32\ \frac{(s-1)(s+10)}{(s+1)(s-10)}.$$