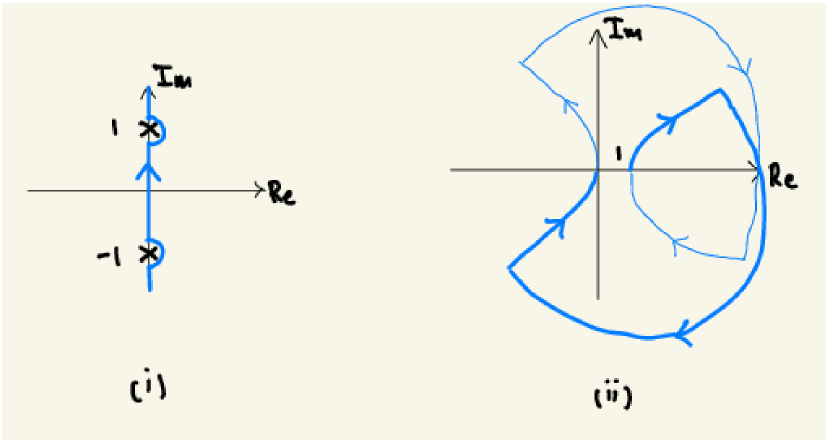
4F1 Solutions 2022

1. (a) (i) $k_i = 0$. The characteristic equation is: $s^2 + k_d s + k_p + 1$. A necessary and sufficient condition for closed-loop stability is: $k_d > 0$ and $k_p > -1$.

(ii) $k_i \neq 0$. The characteristic equation is: $s^3 + k_d s^2 + (k_p + 1)s + k_i$. A necessary and sufficient condition for closed-loop stability is: $k_d > 0$, $k_p > -1$, $k_i > 0$ and $k_d(k_p + 1) > k_i$. [20%]

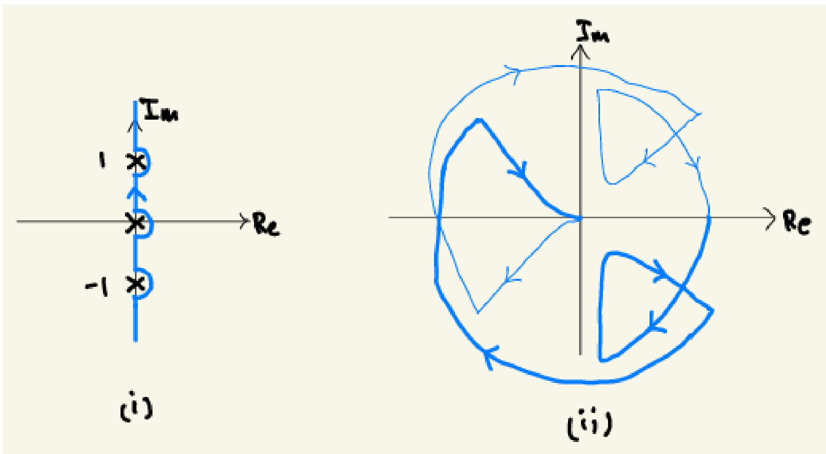
(b)



[15%]

(iii) Closed-loop stable for $k > 0$; 2 RHP poles for $-1 < k < 0$; 1 RHP pole for $k < -1$. [5%]

(c)



[15%]

(iii) 2 RHP poles for $k > 0$; 1 RHP pole for $k < 0$. [5%]

(d) (i) Since $K(s)$ can't have any zeros at $s = \pm j$ because of internal stability $S(j) = 0$. Since $K(s)$ has bounded high frequency gain $S(\infty) = 1$. [10%]

(ii) Since $G(s)K(s)$ has at least second order roll-off at high frequency due to $K(s)$ having bounded high frequency gain

$$\int_0^\infty \ln |S(j\omega)| \, d\omega = 0.$$

Hence there must be a frequency ω_0 such that $|S(j\omega_0)| > 1$. [15%]

(iii) Solving for $K(s)$ algebraically from the given $S(s)$ gives

$$1 + G(s)K(s) = \frac{(s+1)^2}{s^2+1}$$

which implies $K(s) = 2s$, which is a differentiator. It stabilises $G(s)$ but does not have bounded high frequency gain. Note also that

$$|S(j\omega)| = \frac{|1-\omega^2|}{\omega^2+1} \leq 1$$

for all ω and hence Part (d)(ii) no longer holds if the high frequency gain requirement on $K(s)$ is removed. [15%]

Assessor's comment. The least popular question. Almost all candidates successfully completed 1(a). Most candidates managed to produce correct Nyquist diagrams in 1(b)(ii) but less so for 1(c)(ii). Part (d) was well done except for (d)(iii) with few correctly checking the magnitude bound and recognising that a non-proper controller was needed.

2 (a) If no current enters the inverting terminal, the current through Z_1 equals the current through Z_2. Hence, in the Laplace domain,

$$\frac{\bar{v}_1 - \bar{v}}{Z_1} = \frac{\bar{v} - \bar{v}_2}{Z_2}$$

from which the result follows. [10%]

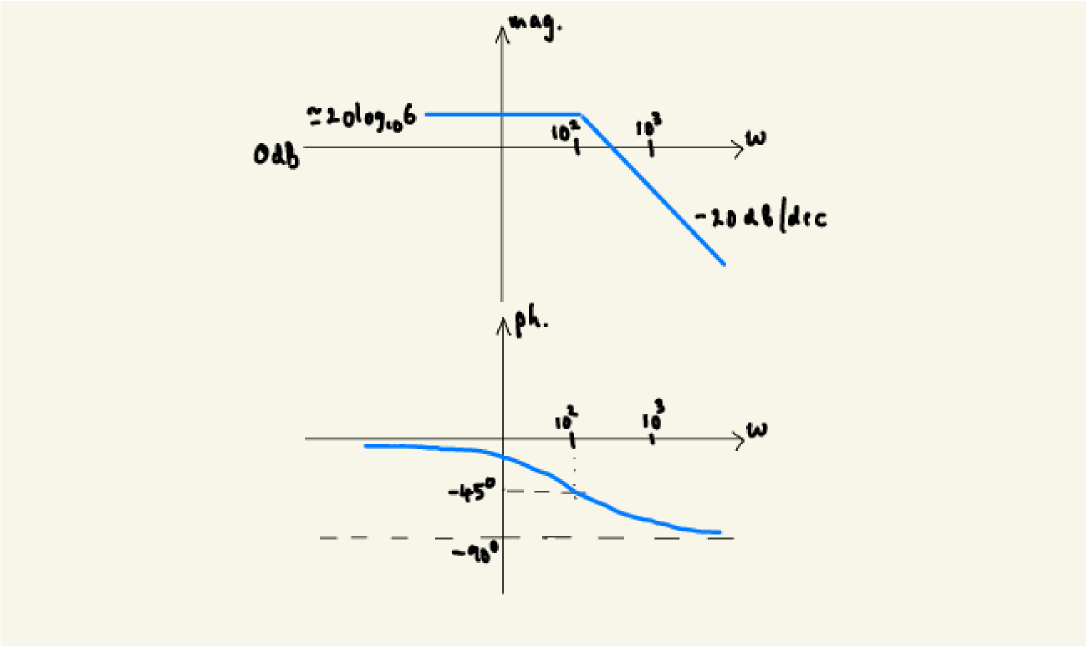
(b) Dividing the equation in (a) by Z_1 gives

$$\bar{v}_2 + \frac{Z_2}{Z_1}\bar{v}_1 = \frac{Z_1 + Z_2}{Z_1}\bar{v}.$$

which is the equation satisfied at the summing junction in the block diagram. [15%]

(c) The transfer function relating $-\bar{v}_2$ to $\bar{v}_1$ is given by:

$$6\frac{1000}{10s + 1001}$$



[15%]

(d) Let $L(s) = Z_1 G/(Z_1 + Z_2)$ and $T = L/(1 + L)$. Then

$$\text{robust stability} \Leftrightarrow |T(j\omega)| < \frac{1}{h(\omega)}$$

for all ω. [15%]

(e) Note that

$$T(j\omega) = \frac{1000}{10j\omega + 1001}$$

which is less than one in magnitude for all ω whereas $h(j\omega)^{-1} > 1$ for all omega. Hence the op amp circuit is robustly stable. [15%]

(f) (i)

$$\frac{Z_1}{Z_1 + Z_2} = \frac{s + 300}{21s + 700} = \frac{1}{7}\frac{s + 300}{3s + 100}$$

which is a lag compensator with maximum lag at $\omega = 100$ rad/sec. [10%]

(ii)

$$L = \frac{s + 300}{3s + 100}\frac{1000}{10s + 1}$$

so the phase is between 0^o and -180^o and hence the loop is stable by the Nyquist stability criterion. [10%]

(iii) Note that $|L(j100)| = 1.00$ and $\angle L(j100)| = -143^o$ so PM=37^o. This will make $T(j100)$ comfortably larger than one. In fact: $T(j100) = 1.5788$ whereas $h(100)^{-1} = 1.407$ so the op amp circuit is not robustly stable. [10%]

Assessor's comment. 2(a) was easily done by the majority of candidates but the verification of the block diagram in 2(b) received convoluted attempts with many candidates not appreciating that the question just reduces to checking that the equation at the summing junction is satisfied. In the remaining question parts most candidates understood well what was required and generally marks were lost from inaccuracies rather than misunderstandings.

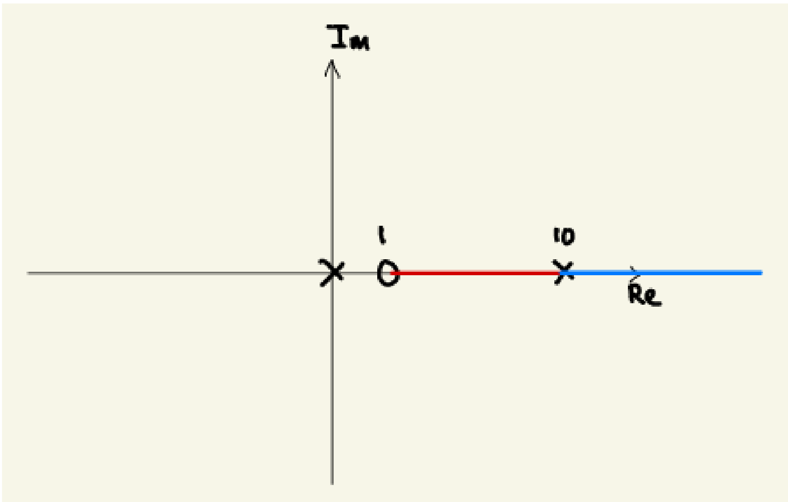
3 (a) (i)

$$G(s) = G_m(s)B_p(s)B_z(s) = \frac{100(1+s)^2}{s(10+s)^2}\,\frac{10+s}{10-s}\,\frac{1-s}{1+s}$$

[10%]

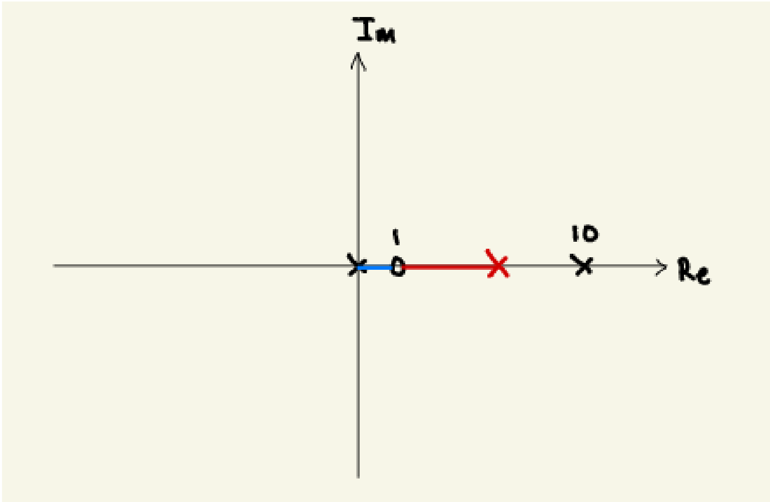
(ii) The loop gain will need to be larger than one around the frequency of the RHP pole ($\omega = 10$ rad/sec) and smaller than one around the frequency of the RHP zero ($\omega = 1$ rad/sec). It will also need to be larger than one at $\omega = 0$ because of the pole at the origin. A challenging loop shape! [15%]

(b) (i) With no poles or zeros added in the right half plane there are branches of the root-locus trapped in the RHP as shown in the figure, either the blue or the red line according to the sign of the feedback gain. Additional RHP zeros don't improve the situation. Breakaway points only change the number of poles on the axis by an even number.



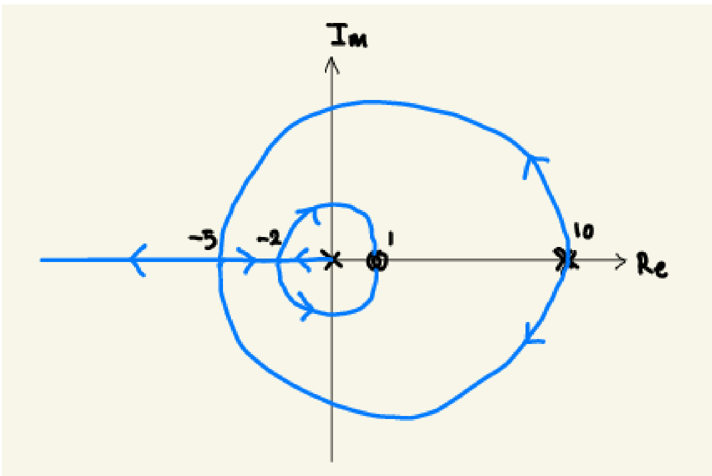
[10%]

(ii) With a single pole added to the right of the zero it is now possible for the pair of poles to break away and move towards the LHP. But consider the real axis rule. If the two poles move together then the pole at the origin moves to the right (blue line in the figure) otherwise the red line is on the root-locus. Again the situation is not improved by breakaway points.



[15%]

(c) (i)



[15%]

(ii) We find

$$G_1(-2) = \frac{(-3)^2}{-2(-12)^2} = \frac{-1}{32}.$$

thus the required feedback gain is $k = 32$. [10%]

(d) (i) To find a stabilising $K(s)$ solve:

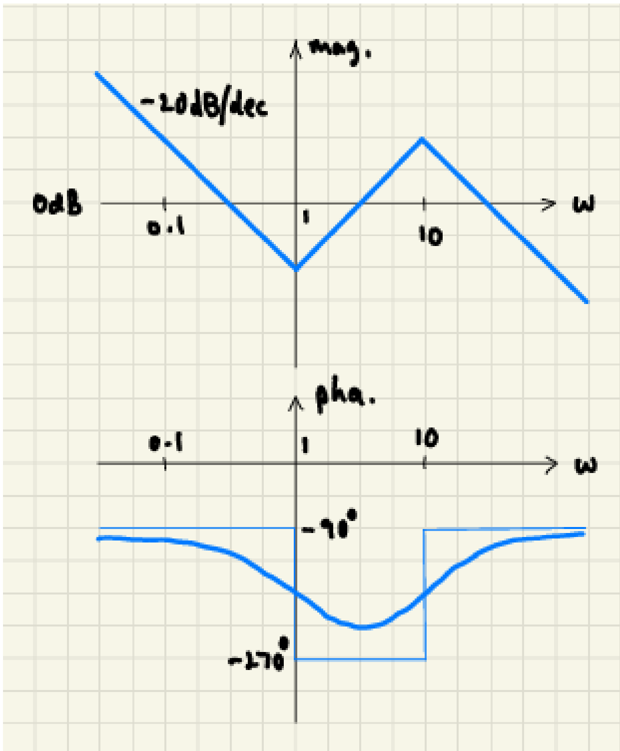
$$G(s)K(s) = 32\,\frac{(s-1)^2}{s(s-10)^2}$$

which gives

$$K(s) = 0.32\,\frac{(s-1)(s+10)}{(s+1)(s-10)}.$$

There are no RHP pole-zero cancellations between $G(s)$ and $K(s)$ so this is a stabilising controller. [10%]

(ii)



Note loop shape of form predicted in (a)(ii). [15%]

Note (not required by candidates): the choice of cart velocity as the only measurement makes the feedback control very difficult (though it does serve as an excellent illustration of the difficulties of non-conventional loop-shapes). State feedback makes this problem a lot easier - see 3F2 lab experiment.

Assessor's comment. A popular question with some parts consistently well done but others causing difficulties. 3(a)(ii) was often poorly answered with candidates not grasping that a non-conventional loop shape was implied. A majority of candidates found 3(b)(ii) difficult though there were some nice solutions. Part (c) was generally well done, and (d) mostly well done.

Figure 1: Computer plot for 3(c)(i).

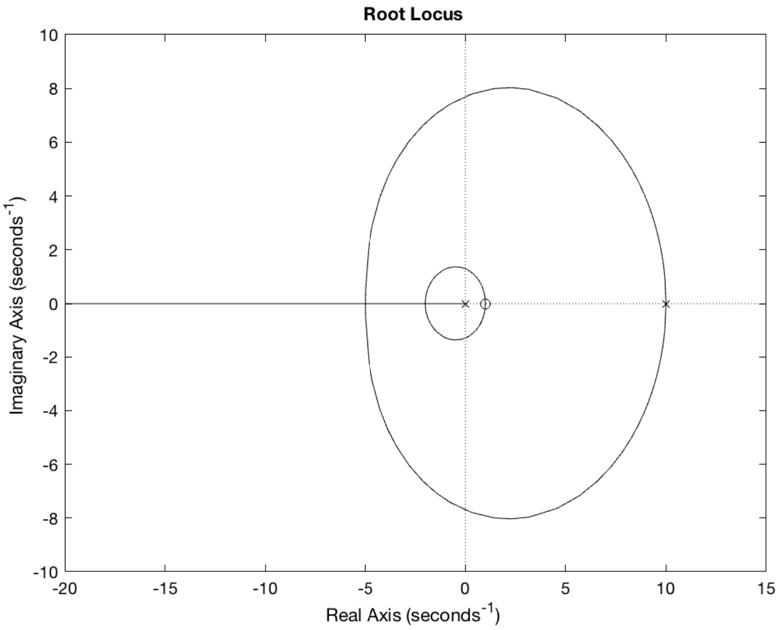
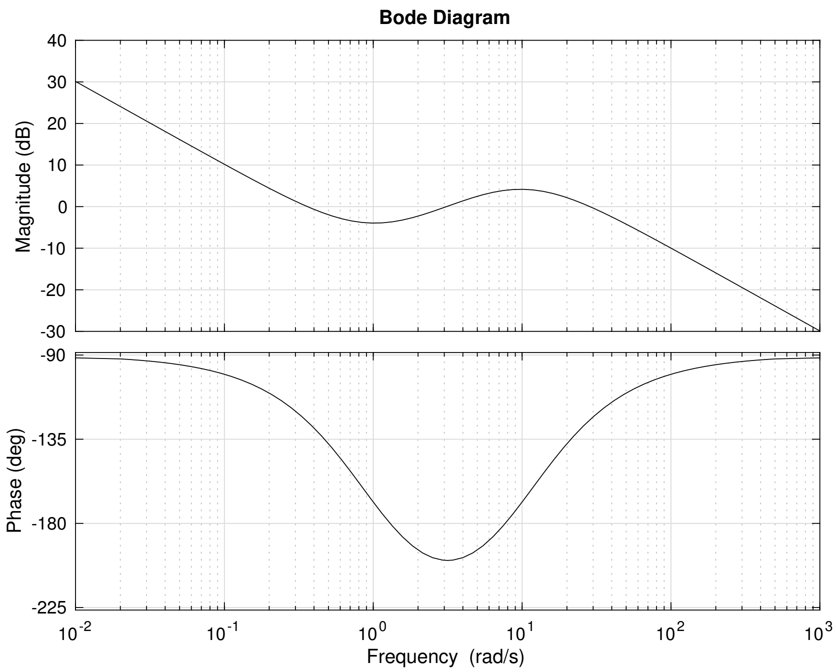


Figure 2: Computer plot for 3(d)(ii).



M.C. Smith, 5 May 2022